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Upper Bounds for Covering Minima of Convex Bodies

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Introduction

The *covering minima* are a sequence of d functionals on the space of all d dimensional convex bodies, which depend on a fixed lattice. They were first introduced by Kannan and Lovász ([14]) and were shown to interpolate between two already known functionals – the reciprocal of the *lattice width*, and the *covering radius*. The lattice width quantifies the minimal number of parallel copies of a lattice hyperplane that intersect the convex body, or in other words – how "flat" a convex body is with respect to the lattice. A celebrated result by Khinchine ([16]) is that a *hollow* convex body, ie one that does not contain interior lattice points, has to be rather flat – there exists a *Flatness Constant*, depending only on the dimension of the body, such that the lattice width of a hollow convex body can be at most this constant. This result was used by Lenstra ([18]) to show that integer linear programming in a fixed dimension admits a polynomial time algorithm. In the paper where covering minima are introduced, Kannan & Lovász exploit this sequence of functionals to obtain the first polynomial upper bound on the Flatness Constant. Values of covering minima that do not coincide with the lattice width or the covering radius are not known for many bodies. Furthermore, even though there exist algorithms for computing both the lattice width (Charrier, Feschet & Buzer [3]) and the covering radius (Cslovjecsek, Malikiosis, Naszódi & Schymura [7]) for rational polytopes, there is no known algorithm for finding any other covering minimum of a rational polytope.

It is easy to see that the maximal covering radius of a lattice d-polytope is d, and that the only maximizers are unimodular simplices. As all unimodular simplices are hollow, Codenotti, Santos & Schymura propose the problem of finding the maximal covering radius of a non-hollow lattice polytope. This problem was shown to be equivalent to calculating the covering minima of simplices $T_d := \operatorname{conv}(-\mathbb{1}_d, e_1, \ldots, e_d)$ (Codenotti, Santos & Schymura [5]), and these problems were solved by the authors in dimensions up to 3. Surprisingly, this was not done on the side of calculating the covering minima, which is seemingly the simpler side of this equivalence. The conjectured non-hollow lattice polytope maximizers of covering radius are the simplices T_d and direct sums of translates of these. The behaviour of the lattice width and covering radius with respect to direct sums is known (see eg. Codenotti & Santos [6] and Codenotti, Santos & Schymura [5]), but is not known for the general covering minima. It is easy to see that values of covering minima of T_d are at least the conjectured values, therefore with this in mind, we are interested in obtaining upper bounds on covering minima. In general, lower bounds of covering minima have been studied: already by Kannan & Lovász [14] with respect to the lattice point enumerator, as well as by Codenotti, Santos & Schymura [5] and Merino & Schymura [9] with respect to the volume of the body, in the context of the Covering Product Conjecture motivated by the conjecture of Makai Jr. [20] and the fundamental theorems of Minkowski [22]. Known upper bounds are due to Kannan & Lovász, who gave upper bounds on covering minima that involve Minkowski's successive minima, which Henk, Schymura & Xue [11] strenghtened by replacing the successive minima with packing minima.

In this thesis, we aim find upper bounds for the covering minima of convex bodies, as well as explain how covering minima interact with direct sums.

The thesis is divided into seven sections. In Section 1, we give some general preliminaries on Convex and Lattice Geometry, as well as a detailed introduction to the lattice width and covering radius of a convex body. In Section 2, we will introduce covering minima and present some of their properties and connections to the lattice width and covering radius, and we will present all known values of covering minima of specific convex bodies. In Section 3, we show the maximizers of the covering radius in the family of lattice polytope, and present known results and conjectures for maximizers in the family of non-hollow lattice polytopes. The known upper bounds on general covering minima will be presented in Section 4.

In Section 5, we give an upper bound on covering minima of convex bodies, which depends on the lower dimensional covering minima of projections and intersections of the convex body with respect to certain linear subspaces. Furthermore, we will give a formula for covering minima of the direct sum of two convex bodies, which connects the two results from [6] and [5] on lattice width and covering radius. Utilizing this general bound, we give an upper bound for the covering minima of simplices T_d. In Section 6, we prove another upper bound on covering minima of convex bodies, which involves only the covering radii of intersections of the body with coordinate subspaces. We then use this bound to give a second upper bound on covering minima of T_d. Furthermore, we show that this upper bound is going to be sharp for all covering minima of a special family of convex bodies, which includes all of the convex bodies for which values of all covering minima are known. In Section 7, we compare the three upper bounds on covering minima of convex bodies – the ones from Sections 4, 5 and 6. Throughout this thesis, the running examples are exactly the lattice polytopes for which values of covering minima are known, and we compare the given bounds on these, as well as our simplices of interest T_d. For all examples, at least one of our bounds is better than the known ones. In the case of simplices T_d, the bound from Section 5 is always slightly better than the known bounds, and in general performs better for small values of i compared to d, whereas the bound from Section 6 is slightly worse for small values of i, but significantly better for big values of i compared to d. For example, for $i = d - 1$, the conjectured value of the covering minimum is $\frac{d-1}{2}$, the known bounds are valued around d, and the bound from Section 6 is valued below $\frac{d}{2}$.

1 Mathematical preliminaries

For a ring R, we denote by $GL_d(R)$ the general linear group of order d, ie all invertible $d \times d$ matrices over R.
For a linear subspace $L \leqslant \mathbb{R}^d$, by π_L we denote the orthogonal projection of the space $\mathbb{R}^d$ to the subspace L, and by $L^\perp$ we denote its orthogonal complement.
By $\mathcal{A}_i(\mathbb{R}^d)$ we denote the family of all i dimensional affine subspaces of $\mathbb{R}^d$.
By $\mathbb{1}_d$ and $\mathbb{0}_d$ we denote the all ones and all zeroes vectors in $\mathbb{R}^d$, and with $e_1, \ldots, e_d \in \mathbb{R}^d$ we denote the standard basis vetors.
We will assume knowledge of basic notions on polytopes, a good reference for this is [25]. We denote the standard simplex of dimension d to be $S_d := \operatorname{conv}(\mathbb{0}_d, e_1, \ldots, e_d)$, the d-hypercube $C_d := [-1,1]^d$ and the d-crosspolytope $C_d^* := \operatorname{conv}\{\pm e_i \mid i \in [d]\}$. With T_d we denote the simplex $\operatorname{conv}(-\mathbb{1}_d, e_1, \ldots, e_d)$. These simplices will be crucial in this thesis, and will be much more discussed in Subsection 3.1.

1.1 Convex Bodies

Definition 1.1. A subset $K \subseteq \mathbb{R}^d$ is a *convex body* if it is convex, compact and full dimensional, ie $\dim(\operatorname{aff}(K)) = d$. We denote the family of all convex bodies in $\mathbb{R}^d$ by $\mathcal{K}^d$.

Definition 1.2. Let $K \in \mathcal{K}^d$ be a convex body. Its *polar body* is

$$K^* := \left\{ f \in (\mathbb{R}^d)^* \mid fx \leqslant 1, \text{ for all } x \in K \right\}.$$

Definition 1.3. We say that a convex body $S \in \mathcal{K}^d$ is *o-symmetric* if its symmetric around the origin, ie $S = -S$.

Definition 1.4. Let $\mathbb{R}^d = V \oplus W$ be a decomposition of $\mathbb{R}^d$ into subspaces of dimensions $\dim(V) = l$, $\dim(W) = d - l$, $K \subseteq V$ and $L \subseteq W$ convex bodies, full dimensional in their respective subspaces, that contain the origin. We define the *direct sum* of these convex bodies as

$$K \oplus L := \{\lambda x + (1-\lambda)y \mid x \in K,\ y \in L\ ,\ \lambda \in [0,1]\}.$$

Remark 1.5. Notice that we don't reqire $\mathbb{0}_d$ to be in the relative interior of K and L, which differs from the standard definition of the direct sum of polytopes which agrees with the face lattice. Since when we talk about polytopes, we don't comment on combinatorial types, this definition is more suitable because our results work with this more general construction.

1.2 Lattices

Definition 1.6. Let $f_1, \ldots, f_d \in \mathbb{R}^d$ be linearly independent. The set $\Lambda = \operatorname{span}_{\mathbb{Z}}(f_1, \ldots, f_d)$ is called a (d-dimensional) *lattice*. We denote the family of all lattices in $\mathbb{R}^d$ with $\mathcal{L}^d$. We refer to the elements of Λ as *lattice points*.

From this definition, we can see lattices as full dimensional linear images of $\mathbb{Z}^d$, ie $\mathcal{L}^d = \{A\mathbb{Z}^d \mid A \in GL_d(\mathbb{R})\}$. An equivalent definition of a lattice is as a discrete additive subgroup of full dimension of $\mathbb{R}^d$. This equivalence is not a trivial observation, and for a general background on lattices as well as convex bodies, we refer to [10].

Convention 1.7. When the lattice is omitted in any notation further, it is assumed to be $\mathbb{Z}^d$ for the appropriate dimension d.

Definition 1.8. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. We say that $a \in \Lambda$ is *primitive* if there does not exist $b \in \Lambda$ and $n \in \mathbb{N}_{>1}$ such that $a = nb$.

Definition 1.9. Let $A \in GL_d(\mathbb{R})$ be an invertible matrix, and $\Lambda = A\mathbb{Z}^d$ a lattice. The *determinant* of the lattice Λ is $\det \Lambda := |\det A|$.

Definition 1.10. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. Then, every set of vectors $f_1, \dots, f_d \in \mathbb{R}^d$ such that $\Lambda = \operatorname{span}_{\mathbb{Z}}(f_1, \dots, f_d)$ is called the *basis* of the lattice Λ.

Definition 1.11. Let $v_0, \dots, v_d \in \mathbb{R}^d$ be affinely independent. The *standard half-open parallelepiped* spanned by these points is the set

$$\Pi(v_0 \dots, v_d) := \left\{ v_0 + \sum_{i=1}^{d} \lambda_i v_i \mid 0 \leqslant \lambda_i < 1 \text{ for all } i \in [d] \right\}.$$

Additionally, if $v_0 = \mathbb{0}_d$ and $v_1, \dots, v_d$ is a basis of Λ, we call this standard half-open parallelepiped a *fundamental domain* of the lattice Λ.

Proposition 1.12. *Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and $v_0, \dots, v_d \in \Lambda$ affinely independent. Then*

$$\Pi(v_0, \dots, v_d) + \Lambda = \mathbb{R}^d.$$

Additionally, if $\Pi(v_0, \dots, v_d)$ is a fundamental domain, every point $x \in \mathbb{R}^d$ can be uniquely represented as $x = p + a$, where $p \in \Pi(v_0, \dots, v_d)$ and $a \in \Lambda$.

Definition 1.13. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. A linear map $A : \mathbb{R}^d \to \mathbb{R}^d$ is a *unimodular transformation* if its restriction to Λ is a bijection to Λ.

Definition 1.14. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. We say $P \subseteq \mathbb{R}^d$ is a *lattice polytope* if it is a polytope whose vertices are lattice points. Additionally, P is a *rational polytope* if there exists $n \in \mathbb{N}$ such that nP is a lattice polytope, ie P has vertices in $\mathbb{Q}^d$.

Definition 1.15. Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. We say that $S \subseteq \mathbb{R}^d$ with vertices $v_0, \dots, v_d \in \mathbb{R}^d$ is an *unimodular simplex* if $v_1 - v_0, \dots, v_d - v_0$ is a basis for Λ.

One could prove that A being a unimodular transform is equivalent to it sending a basis of the lattice Λ to a basis of the lattice Λ. More specifically, if we fix a basis of Λ, which is by definition also a basis of $\mathbb{R}^d$, and view this map as a matrix in $GL_d(\mathbb{R})$, we can see that this is exactly the notion of a *unimodular matrix* ie matrix in $GL_d(\mathbb{Z})$. Similarly, every unimodular simplex with respect to the lattuce $\mathbb{Z}^d$ is going to be $A \cdot S_d$, where $A \in GL_d(\mathbb{Z})$.

Proposition 1.16. *Let $\Lambda \subseteq \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$ a lattice simplex with vertices $v_0, \ldots, v_d \in \Lambda$. Then, S is a unimodular simplex if and only if $\Pi(v_0, \ldots, v_d) \cap \Lambda = \{v_0\}$.*

Definition 1.17. Let $\Lambda \in \mathcal{L}^d$ be a lattice and $L \leqslant \mathbb{R}^d$ a linear subspace. We say that L is a *rational subspace* of $\mathbb{R}^d$ with respect to Λ if it has a basis consisting of vectors in the lattice Λ.

Proposition 1.18. *Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and $L \leqslant \mathbb{R}^d$ a linear subspace. Then, L is a rational subspace iff $\Lambda \cap L$ and $\pi_L(\Lambda)$ are lattices.*

Definition 1.19. Let $\Lambda \in \mathcal{L}^d$ be a lattice. Its *dual lattice* is

$$\Lambda^* := \left\{ f \in (\mathbb{R}^d)^* \mid fx \in \mathbb{Z}, \text{ for all } x \in \Lambda \right\}.$$

Specifically, $(\mathbb{Z}^d)^* = \mathbb{Z}^d$.

Definition 1.20. Let $\mathbb{R}^d = V \bigoplus W$ be a decomposition of $\mathbb{R}^d$ into subspaces of dimensions $\dim(V) = l$, $\dim(W) = d - l$ and $\Lambda \subseteq V$, $\Gamma \subseteq W$ lattices. We define the *direct sum* of these lattices as the lattice:

$$\Lambda \bigoplus \Gamma := \{a + b \mid a \in \Lambda,\ b \in \Gamma\}.$$

1.3 Successive Minima

Definition 1.21. For an o-symmetric convex body $S \in \mathcal{K}^d$, a lattice $\Lambda \in \mathcal{L}^d$ and $i \in [d]$, we define Minkowski's *successive minima* as

$$\lambda_i(S, \Lambda) := \min \{\lambda \geqslant 0 \mid \dim(\operatorname{span}_{\mathbb{R}}\{\lambda S \cap \Lambda\}) \geqslant i \}.$$

Remark 1.22. Notice that from the definition of successive minima, for every o-symmetric $K \in \mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$, $\lambda_1(K, \Lambda) \leqslant \cdots \leqslant \lambda_d(K, \Lambda)$. Additionally, notice that if $L \in \mathcal{K}^d$ is o-symmetric and $L \subseteq K$, for every $i \in [d]$, $\lambda_i(L, \Lambda) \geqslant \lambda_i(K, \Lambda)$.
If $\lambda > 0$, then for all $i \in [d]$, $\lambda_i(\lambda K, \Lambda) = \frac{1}{\lambda}\lambda_i(K, \Lambda)$.

Example 1.23. Notice that the hypercube C_d contains d linearly independent lattice vectors, for example $e_1, \ldots, e_d$, and therefore $\lambda_d(C_d) \leqslant 1$. We can also notice that all lattice points in C_d have coordinates 0,1 or -1, and the only one that is in the interior is $\mathbb{0}_d$. Therefore, for any $0 < \lambda < 1$, the only lattice point in λC_d is $\mathbb{0}_d$, and therefore $\lambda_1(C_d) \geqslant 1$. From these two, and monotonicity of successive minima, we get $\lambda_1(C_d) = \cdots = \lambda_d(C_d) = 1$.

Example 1.24. Similarly to previous example, the crosspolytope C_d^* contains $e_1, \dots, e_d$, ie $\lambda_d(C_d^*) \leqslant 1$. Since $C_d^* \subseteq C_d$, we can conclude $\lambda_i(C_d^*) \geqslant 1$ for all $i \in [d]$. From monotonicity of successive minima, $\lambda_1(C_d^*) = \cdots = \lambda_d(C_d^*) = 1$.

Example 1.25. Notice that for any convex body $K \in \mathcal{K}^d$, $K - K$ is an o-symmetric convex body, and therefore its successive minima are well defined.
To see what the successive minima of $S_d - S_d$ is, notice that $e_1, \dots, e_d \in S_d - S_d$, hence $\lambda_d(S_d - S_d) \leqslant 1$. Vertices of $S_d - S_d$ are a subset of the set $\{v+w \mid v \in V(S_d),\ w \in V(-S_d)\}$, all of them are 0-1 vectors. Therefore, $S_d - S_d \subseteq C_d$. In the same manner as before, we conclude $\lambda_1(S_d - S_d) = \cdots = \lambda_d(S_d - S_d) = 1$.

Along with introducing the successive minima Minkowski ([22]) began the field of geometry of numbers, with the following result:

Theorem 1.26. *(Minkowski's First Fundamental Theorem) Let $S \in \mathcal{K}^d$ be an o-symmetric body and $\Lambda \in \mathcal{L}^d$ a lattice. Then:*

$$\lambda_1(S, \Lambda) \leqslant 2 \left(\frac{\det(\Lambda)}{\operatorname{vol}(S)} \right)^{\frac{1}{d}}.$$

This is not the most natural formulation of this theorem, but it's the one that suits our purposes the best. The more intuitive way of understanding this theorem is that it says that a $o-$symmetric body with volume at least $2^d \det(\Lambda)$ has to contain a non-zero lattice point.

We can see that Minkowski's First Fundamental Theorem can also be written as

$$\lambda_1(S, \Lambda)^n \operatorname{vol}(S) \leqslant 2^d \det(\Lambda).$$

The natural strenghtening of this theorem was also given by Minkowski.

Theorem 1.27. *(Minkowski's Second Fundamental Theorem) Let $S \in \mathcal{K}^d$ be an o-symmetric body and $\Lambda \in \mathcal{L}^d$ a lattice. Then:*

$$\lambda_1(S, \Lambda) \cdot \dots \cdot \lambda_d(S, \Lambda) \operatorname{vol}(S) \leqslant 2^d \det(\Lambda).$$

1.4 Lattice Width

Definition 1.28. Let $K \subseteq \mathbb{R}^d$ be a convex body and $\Lambda \subseteq \mathbb{R}^d$ a lattice, and $f \in (\mathbb{R}^d)^*$ a linear functional. The *width* of K with respect to the linear functional f is

$$\omega(K, f) := \max_{x,y \in K} |fx - fy|.$$

The *lattice width* of K with respect to the lattice Λ is

$$\omega_\Lambda(K) := \min_{f \in \Lambda^* \setminus \{0\}} \omega(K, f).$$

This notion is motivated by the Eucledian width of a convex body, but has an essentially different behaviour. The Eucledian width is the minimum width when considering only the normalized functionals, specifically all the functionals corresponding to points of the sphere $\mathbb{S}^{d-1}$. In the lattice width case, the functionals we are considering correspond to real vectors of different lengths, but with integer coordinates, so the lattice width encapsulates more number theoretical information then the purely metric information that the Eucledian width encapsulates.

Remark 1.29. Notice that for any full dimensional lattice polytope P, its lattice width has to be a positive integer. This follows from the fact that every functional $f \in \Lambda^* \backslash \{0\}$ takes integer values on all vertices of P, and it has to be maximized and minimized in at least one vertex. It cannot be zero, because that would imply that P lies in the hyperplane defined by the width achieving direction.
Moreover, in the class of lattice polytopes, we can view the lattice width as the maximal number of lattice hyperplanes in any fixed direction intersecting it, minus 1.

Remark 1.30. Notice that we can restrict the search for width achieving directions to primitive non-zero lattice functionals, since if $a = nb$ for $a, b \in \Lambda$ and $n \in \mathbb{N}_{>1}$, for every $x \in \mathbb{R}^d$, $ax = n(bx)$, so $\omega(K, a) = n\omega(K, b)$. Specifically, $\omega(K, b) < \omega(K, a)$ and a cannot be a width achieving direction.

Example 1.31. The lattice width of the standard unimodular simplex S_d with respect to the lattice $\mathbb{Z}^d$ is 1. Since S_d is a lattice polytope, $\omega(S_d)$ is a positive integer, specifically $\omega(S_d) \geqslant 1$. Observe the non zero lattice functional f that just gives back the first coordinate of a point. Then, $fe_1 - f\mathbb{0}_d = 1$, and therefore $\omega(S_d) \leqslant \omega(S_d, f) \leqslant 1$.

Remark 1.32. Notice that the lattice width is positively homogeneous, ie for every $\lambda > 0$, $\omega_\Lambda(\lambda K) = \lambda\omega_\Lambda(K)$.

Definition 1.33. We say that a convex body $K \in \mathcal{K}^d$ is *hollow* with respect to the lattice Λ if it doesn't have interior lattice points, ie $\operatorname{int}(K) \cap \Lambda = \varnothing$.

Remark 1.34. In literature, the notion of a convex body being *hollow* is sometimes also refered to it as being *lattice-free*.

An important result regarding the lattice width is the *Flatness Theorem*, which states that a hollow convex body cannot be arbitrarily wide.

Theorem 1.35 (Flatness Theorem, [16])**.** *There exists a constant $Flt(d)$ depending just on the dimension d such that for any hollow convex body $K \in \mathcal{K}^d$, the following holds:*

$$\omega(K) \leqslant Flt(d).$$

Khinchine gave an upper bound for $Flt(d)$ in [16] of order of magnitude $n!$. The first polynomial bound on this value, $O(n^2)$, was given by Kannan and Lovász in [14], using the notion of covering minima, which are the focus of this thesis. This is still far off from the expected value $O(d)$, which is the best one could hope to get since dS_d is a hollow d-polytope of lattice width d with respect to $\mathbb{Z}^d$. Recently, Reis and Rothvoss ([23]) gave the best known upper bound on $Flt(d)$ being $O(d \log^3 d)$, also using certain notions from the Kannan and Lovász paper. Moreover, the only known exact values for $Flt(d)$ are for $d = 1, 2$, where the first case just states that no segment of length bigger than 1 is hollow, and the second is a result by Hurkens ([12]), where he shows $Flt(2) = 1 + \frac{2}{\sqrt{3}}$.

The *integer linear programming* problem is the question of deciding if a given system of linear inequalities with integer coefficients has an integer solution. There is no known algorithm in polynomial time with respect to the lenght of the input, and the problem phrased like that is proven to be NP-complete.

The existance of the Flatness constant was used by Lenstra in [18] to find a polynomial algorithm for integer linear programming in fixed dimension. Moreover, the algorithm would either efficiently find a solution, or find a width direction and reduce the problem to a bounded number of lower dimensional problems.

The following proposition is folklore and shows a connection between the lattice width and successive minima.

Proposition 1.36. *Let $K \in \mathcal{K}^d$ be a convex body and $\Lambda \in \mathcal{L}^d$ a lattice. Then,*

$$\omega(K, \Lambda) = \lambda_1((K - K)^*, \Lambda^*).$$

Proof. From the definition of successive minima 1.21, we can notice that the first successive minimum can be seen as the smallest scaling of the convex body that contains a non-zero lattice point. Specifically, in this example:

$$\lambda_1((K - K)^*, \Lambda^*) := \min \{\lambda > 0 \mid \lambda(K - K)^* \cap \Lambda^* \neq \{\mathbb{0}_d\}\}.$$

Denote by $\lambda_1^* := \lambda_1((K - K)^*, \Lambda^*)$, and let $f \in (\lambda_1^*(K - K)^* \cap \Lambda^*) \backslash \{\mathbb{0}_d\}$. By definition of the polar body for all $p \in K - K$, $fp \leqslant \lambda_1^*$, which is equivalent to $fx - fy \leqslant \lambda_1^*$ holding for all $x, y \in K$. Since this holds for all $x, y \in K$, it is equivalent to $|fx - fy| \leqslant \lambda_1^*$ holding for all $x, y \in K$.
This would now imply that

$$\omega_\Lambda(K, f) := \max_{x,y \in K} |fx - fy| \leqslant \lambda_1^*,$$

and from minimality of λ_1^* we can conclude $\omega_\Lambda(K, f) = \lambda_1^*$. Moreover, from minimality of λ_1^* we can also notice that

$$\lambda_1^* = \omega_\Lambda(K, f) = \min_{g \in \Lambda^* \backslash \{0\}} \omega_\Lambda(K, g) =: \omega_\Lambda(K).$$

$\square$

Remark 1.37. From the previous proof we can also notice that the directions in which the lattice width of a convex body K is achieved are exactly the non-zero lattice points in $\lambda_1((K-K)^*, \Lambda)(K-K)^*$.

Example 1.38. We can now show that $\omega(C_d) = 2$. From Proposition 1.36, we know $\omega(C_d) = \lambda_1((C_d - C_d)^*) = \lambda_1((2C_d)^*) = \lambda_1(\frac{1}{2}C_d^*) = 2\lambda_1(C_d^*) = 2$, by Example 1.24.

Example 1.39. Similar to previous example, $\omega(C_d^*) = 2\lambda_1(C_d) = 2$, by Example 1.23.

In practice, this is the easiest way to algorithmically compute the width of a given convex body. The issue is that to make it efficient, one would have to find the shortest vectors in a lattice, which is NP-hard. A more efficient algorithm for computing the lattice width is given by Charrier, Feschet and Buzer in [3].

1.5 Covering Radius

Definition 1.40. Let $K \in \mathcal{K}^d$ be a convex body and $\Lambda \in \mathcal{L}^d$ a lattice. The *covering radius* of K with respect to the lattice Λ is

$$\mu(K, \Lambda) := \min \left\{ \mu \geqslant 0 \mid \mu K + \Lambda = \mathbb{R}^d \right\}.$$

Example 1.41. The covering radius of the standard cube $C_d = [-1,1]^d$ with respect to the lattice $\mathbb{Z}^d$ is $\frac{1}{2}$. For every $0 < \varepsilon < \frac{1}{2}$, for example the point $\frac{1}{2}\mathbb{1}_d$ will not be contained in $(\frac{1}{2} - \varepsilon)C_d + \mathbb{Z}^d$, therefore $\mu(C_d) \geqslant \frac{1}{2}$. On the other hand, for $x \in \mathbb{R}$, if we denote by $[x]$ the closest integer to x, where $[a + \frac{1}{2}] := a$ for $a \in \mathbb{Z}$, we can see that for any $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$, we can decompose it as $(p_1 - [p_1], \ldots, p_d - [p_d]) + ([p_1], \ldots, [p_d]) \in \frac{1}{2}C_d + \mathbb{Z}^d$, and therefore $\mu(C_d) \leqslant \frac{1}{2}$.

Example 1.42. The covering radius of the d-crosspolytope with respect to $\mathbb{Z}^d$ is $\frac{d}{2}$. Notice that $C_d \subseteq dC_d^*$, since for any vertex of the hypercube, $v \in \{-1,1\}^d$, $v = \sum_{i=1}^{d} \frac{1}{d} dv_i e_i$, and $dv_i e_i$ are vertices of C_d^*. Therefore, $\frac{1}{2}C_d \subseteq \frac{d}{2}C_d^*$, and Example 1.41 tells us that $\frac{1}{2}C_d + \mathbb{Z}^d = \mathbb{R}^d$, so $\mu(C_d^*) \leqslant \frac{d}{2}$.
Now, to prove that $\mu(C_d^*) \geqslant \frac{d}{2}$, we will show that the point $\frac{1}{2}\mathbb{1}_d$ cannot be in $(\frac{d}{2} - \varepsilon)C_d^* + \mathbb{Z}^d$ for any $\frac{d}{2} > \varepsilon > 0$. Assume the countrary, and let $a \in \mathbb{Z}^d$ be such that $\frac{1}{2}\mathbb{1}_d \in a + (\frac{d}{2} - \varepsilon)C_d^*$ and a is such that it has the minimal possible number of strictly positive entries. For $i \in [d]$, let $\alpha_i, \beta_i \geqslant 0$, $\sum_{i=1}^{d} \alpha_i + \sum_{i=1}^{d} \beta_i = 1$ such that $\frac{1}{2}\mathbb{1}_d = a + \sum_{i=1}^{d} \alpha_i(\frac{d}{2} - \varepsilon)e_i - \sum_{i=1}^{d} \beta_i(\frac{d}{2} - \varepsilon)e_i$. Then, if by a_i we denote the i-th coordinate of the point a for all $i \in [d]$, $\frac{1}{2} - a_i = (\frac{d}{2} - \varepsilon)(\alpha_i - \beta_i)$. Summing up all of these inequalities, we get $\frac{d}{2} - \sum_{i=1}^{d} a_i = (\frac{d}{2} - \varepsilon)(\sum_{i=1}^{d} \alpha_i - \sum_{i=1}^{d} \beta_i)$. Since

these coefficients are non-negative, $\sum_{i=1}^{d} \alpha_i - \sum_{i=1}^{d} \beta_i \leqslant \sum_{i=1}^{d} \alpha_i + \sum_{i=1}^{d} \beta_i = 1$, so the right hand side of this equality has to be strictly less than $\frac{d}{2}$, since $\varepsilon > 0$. That means there is at least one strictly positive a_i and without loss of generality, assume $a_1 > 0$. Then, by swapping the coefficients α_1 and β_1, none of the other equations change, and the first one becomes $(\frac{d}{2} - \varepsilon)(\beta_i - \alpha_i) = -(\frac{1}{2} - a_1) = \frac{1}{2} - (-a_1 + 1)$. This means $\frac{1}{2}\mathbb{1}_d \in (-a_1 + 1, a_2, \dots, a_d) + \frac{d}{2} C_d^*$, and since a_1 was strictly positive, $-a_1 + 1$ is not, which contradicts the minimality condition for a.

A more geometric interpretation of the covering radius of a convex body is to be seen in the following proposition, which is folklore and we will prove it for completness.

Proposition 1.43. *Let $K \in \mathcal{K}^d$ be a convex body and $\Lambda \in \mathcal{K}^d$ a lattice. Then its covering radius is the maximal scalar $\mu \geqslant 0$ such that μK admits a hollow translate.*

Proof. Let $\mu := \mu(K, \Lambda)$ and $\mu' := \max\{\lambda \geqslant 0 \mid \lambda K$ admits a hollow translate$\}$. For convenience, suppose $\mathbb{0}_d \in K$, so that for every $\lambda' \leqslant \lambda$ we can claim $\lambda' K \subseteq \lambda K$, since K is convex. We can make this assumption since both of the values μ and μ' are translatory invariant.

First, let's prove $\mu \geqslant \mu'$. Let $p \in \mathbb{R}^d$ be such that $p + \mu' K$ is hollow, ie

$$(p + \mu' \operatorname{int}(K)) \cap \Lambda = \varnothing.$$

Then, we claim that $-p \notin \mu' \operatorname{int}(K) + \Lambda$. Suppose the opposite, $-p = \mu' q + a$, wherer $q \in \operatorname{int}(K)$ and $a \in \Lambda$. Then, $-a = p + \mu' q$, and here $-a \in \Lambda$ and $p + \mu' q \in p + \mu' \operatorname{int}(K)$, which is impossible since $p + \mu' K$ is hollow. Now since $p \notin \mu' \operatorname{int}(K) + \Lambda$, so $\mu' \operatorname{int}(K) + \Lambda \neq \mathbb{R}^d$, we can conclude that for every $\varepsilon > 0$, $(\mu' - \varepsilon)K + \Lambda \subseteq \mu' K + \Lambda \neq \mathbb{R}^d$, therefore by letting ε approach zero, we get $\mu' \leqslant \mu$.

Now, let's prove that μK admits a hollow translate, which would imply $\mu \leqslant \mu'$. Similar to the first inequality, we want to take a point $p \in \mathbb{R}^d \backslash (\mu \operatorname{int}(K) + \Lambda)$ and notice that $-p + \mu K$ is hollow, following from the same observations. Therefore, we just need to show that the set $\mathbb{R}^d \backslash (\mu \operatorname{int}(K) + \Lambda)$ is non-empty.
Since μ is minimal such that $\mu K + \Lambda = \mathbb{R}^d$, for every $\varepsilon > 0$, $(\mu - \varepsilon)K + \Lambda \neq \mathbb{R}^d$. Let $F \subseteq \mathbb{R}^d$ be a fundamental domain of the lattice Λ. For any point $x \in \mathbb{R}^d \backslash \{(\mu - \varepsilon)K + \Lambda\}$, by Proposition 1.12, there exist unique representation of x as a sum of a point in $x_\varepsilon \in F$ and a point $a \in \Lambda$. Since $x_\varepsilon + a \in \mathbb{R}^d \backslash \{(\mu - \varepsilon)K + \Lambda\}$, we can conclude $x_\varepsilon \in F \backslash \{(\mu - \varepsilon)K + \Lambda\}$. The closure of a fundamental domain is compact, so $\{x_\varepsilon\}_{\varepsilon > 0}$ has a limit point $x \in \operatorname{cl}(F)$. Then we can say that $x \notin (\mu - \varepsilon)K + \Lambda$ for every $\varepsilon > 0$, and therefore is not in $\mu \operatorname{int}(K) + \Lambda$. ☐

Example 1.44. The covering radius of the unimodular simplex $S_d = \operatorname{conv}(\mathbb{0}_d, e_1, \ldots, e_d)$ with respect to the lattice $\mathbb{Z}^d$ is d.
From Proposition 1.43, to prove $\mu(S_d) \geqslant d$ it suffices to show that dS_d is hollow. Since $d\,\operatorname{int}(S_d) = \left\{x \in \mathbb{R}^d \mid x_i > 0,\ \sum_{i=1}^{d} x_y < d\right\}$, there is no integers $x_1, \ldots, x_d$ satisfiying these conditions, ie dS_d is hollow. Since $[0,1]^d \subseteq dS_d$, and $[0,1]^d$ is a fundamental domain, $dS_d + \mathbb{Z}^d \supseteq [0,1]^d + \mathbb{Z}^d = \mathbb{R}^d$, ie $\mu(S_d) \leqslant d$.

The following definition is due to Codenotti, Santos and Schymura ([5]), and encapsulates the ideas seen in the previous proof.

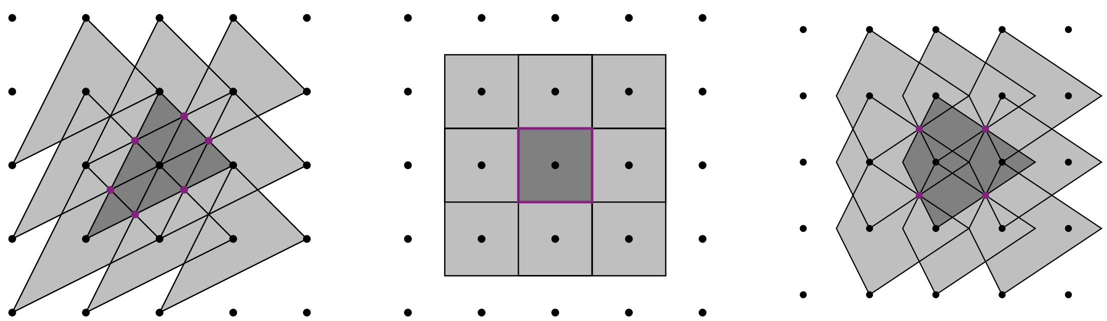
Definition 1.45. Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $\mu := \mu(K, \Lambda)$. A point $p \in \mathbb{R}^d$ is *last covered by* K with respect to Λ if

$$p \notin \mu\,\operatorname{int}(K) + \Lambda.$$

Notice that the second step of this proof was essentially just proving that last covered points always exist.

Remark 1.46. Notice that if for some $\mu \geqslant 0$, since last covered points always exist, ie $\mu(K, \Lambda)\,\operatorname{int}(K) + \Lambda \neq \mathbb{R}^d$, the following two hold:

1. $\mu\,\operatorname{int}(K) + \Lambda \neq \mathbb{R}^d \;\Rightarrow\; \mu \leqslant \mu(K, \Lambda)$,
2. $\mu\,\operatorname{int}(K) + \Lambda = \mathbb{R}^d \;\Rightarrow\; \mu > \mu(K, \Lambda)$.



(a) T_2 (b) C_2 (c) $\operatorname{conv}(-\frac{1}{2}e_1, \frac{3}{2}e_1, -e_2, e_2)$

Figure 1.1: Last Covered Points

Figure 1.1 shows three convex bodies, scaled by their covering radii which are 1, $\frac{1}{2}$ and 1 respectively, shaded in dark gray, as well as the lattice $\mathbb{Z}^2$. Shaded in light gray are some of the lattice translates of the convex bodies, and in red are the last covered points that coincide with the original copy of the convex body. From example (b) we can notice that the set of last covered points coinciding with $\mu(K)K$ does not have to be finite. In these examples we can notice that last covered points will always coincide with at least two lattice translates of $\mu(K)\partial K$, but from example (c) we can also notice that not all intersections of at least two lattice translates of $\mu(K)\partial K$ have to be last covered points, since they can be in the interior of another lattice translate of $\mu(K)K$.

Algorithms for calculating the covering radius of a rational polytope have been explored. Kannan ([13]) reduces the Frobenius coin problem, which is known to be NP-hard with respect to the lenght of the input, to calculating the covering radius of specific rational simplices, and gives the first algorithm for calculating covering radii of rational polytopes in the same paper. This proof is quite technical, and after using the obtained structural results, relies on solving multiple mixed integer linear programs.

The notion of last covered points brings a more geometric viewpoint of the study of the covering radius. By approaching this problem from the last covered points point of view, Cslovjcsek, Malikosis, Naszódi and Schymura [7] gave a more efficient and more easily implementable algorithm for computing the covering radius of a given rational polytope. The algorithm is based on the following lemma, which reduces the problem to a binary search where in each step, one solves a system of linear inequations.

Lemma 1.47. *[7, Lem. 3.1] Let $P = \left\{x \in \mathbb{R}^d \mid a_i^T x \leqslant b_i,\ i \in [m]\right\}$ be a facet description of a polytope $P \subseteq \mathbb{R}^d$ with the origin in the interior, ie $b_i > 0$ for all $i \in [m]$. Then, there exist facet normals $a_{i_1}, \dots, a_{i_{d+1}}$ of P and not necessairily distinct lattice points $z_1, \dots, z_{d+1} \in \mathbb{Z}^d$ such that the system of linear equations*

$$\mu = \frac{a_{i_1}^T(x - z_1)}{b_{i_1}} = \cdots = \frac{a_{i_{d+1}}^T(x - z_{d+1})}{b_{i_{d+1}}}$$

in the variables μ and x has a unique solution $(\bar{\mu}, \bar{p})$, where $\bar{\mu} = \mu(P)$ and $\bar{p}$ is a last covered point by P with respect to $\mathbb{Z}^d$.

To prove this lemma, one takes any last covered point that is contained in the most of the sets from the family $\{F + \mathbb{Z}^d \mid F \text{ is a facet of } P\}$. Then, consider the set of facet normals of the facets containg that last covered point in one of their lattice translates. They show that the affine hull of this set is $\mathbb{R}^d$, and therefore that set has to contain at least $d + 1$ points. The system of linear equations essentially describes getting the observed last covered point as the intersection of the scaled facet translates it is in.

Geometrically, this lemma states that for a polytope P, there exists a last covered point p that is exactly the intersection of some lattice translates of $d+1$ distinct facets of $\mu(P)P$.

On the other hand, last covered points are useful for investigating inclusion maximal convex bodies with a fixed covering radius, which were first studied by Codenotti, Santos and Schymura.

Definition 1.48. Let $K \in \mathcal{K}^d$ be a convex body and $\Lambda \in \mathcal{L}^d$ a lattice. Then, K is called *tight for* Λ if for every convex body $K' \supsetneq K$ we have

$$\mu(K', \Lambda) < \mu(K, \Lambda).$$

The following lemma gives a characterization of all tight convex bodies.

Lemma 1.49. *[5, Lem. 2.5] Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $\mu := \mu(K, \Lambda)$. Then, the following properties are equivalent:*

i) K is tight for Λ.

ii) K is a polytope and for every facet F of K and for every last covered point p,

$$p \in \operatorname{relint}(\mu \cdot F) + \Lambda.$$

iii) K is a polytope and every facet of every hollow translate of μK is non-hollow.

iv) Every hollow translate of $\mu \cdot K$ is an inclusion maximal hollow convex body.

The notion of tightness was furthermore used for finding maximizers for covering radis in the family of non-hollow lattice polytopes in dimensions 2 and 3, which there will be more word on in Subsection 3.1. They did this through the following lemmas and the fact that the covering radius is monotonely decreasing with respect to set inclusion.

Lemma 1.50. *[5, Lem 2.7] Every simplex is tight for every lattice.*

Lemma 1.51. *[5, Lem. 2.8] Let K_1 and K_2 be convex bodies containing the origin and Λ_1 and Λ_2 be lattices. Then, K_1 and K_2 are tight for Λ_1 and Λ_2 respectively iff $K_1 \oplus K_2$ is tight for $\Lambda_1 \oplus \Lambda_2$.*

2 Covering Minima

The functionals on the family of convex bodies that we are the most interested in are *covering minima*, introduced by Kannan and Lovász in [14]. In this section, we will see the definition and some basic properties of covering minima, as well as their connections to the previously mentioned functionals. Furthermore, we will see some examples of convex bodies for which we do know the values of all covering minima.

2.1 Definition and Properties

Definition 2.1. Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $i \in [d]$. The *i-th covering minimum* of K with respect to the lattice Λ is

$$\mu_i(K, \Lambda) := \inf \left\{ \mu \geqslant 0 \mid (\mu K + \Lambda) \cap L \neq \varnothing \text{ for all } L \in \mathcal{A}_{d-i}(\mathbb{R}^d) \right\}$$

Remark 2.2. Notice that if the convex body is of dimension d, the notions of d-th covering minimum and the covering radius are by definition the same, ie for every $K \in \mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$,

$$\mu_d(K, \Lambda) = \mu(K, \Lambda).$$

Remark 2.3. Some properties of covering minima are as follows:

- *(Translation invariance)* Since this definition requires intersections with all affine subspaces of fixed dimension, the covering minima are invariant with respect to translation of the convex body.
- *($GL_d(\mathbb{R})$ invariance)* For $A \in GL_d(\mathbb{R})$, A^{-1} is a bijection on $\mathcal{A}_{d-i}$, therefore $\mu AK + A\Lambda$ intersects all $d-i$ affine subspaces iff $\mu K + \Lambda$ intersects them all, ie $\mu_i(AK, A\Lambda) = \mu_i(K, \Lambda)$.
- *(Monotonicity)* Notice that if some subset of $\mathbb{R}^d$ intersects all i-dimensional affine subspaces, it also intersects all $(i-1)$-dimensional affine subspaces, therefore $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_d$.
- *(Monotonicity with respect to inclusion of convex bodies)* Let $K, K' \in \mathcal{K}^d$ and $K' \subseteq K$. Then for every $\mu \geqslant 0$, $\mu K' + \Lambda \subseteq \mu K + \Lambda$, so $\mu(K, \Lambda) \leqslant \mu(K', \Lambda)$.
- *(Monotonicity with respect to inclusion of lattices)* Let $\Lambda, \Lambda' \in \mathcal{L}^d$ and $\Lambda' \subseteq \Lambda$. Then for every $\mu \geqslant 0$, $\mu K + \Lambda' \subseteq \mu K + \Lambda$, so $\mu(K, \Lambda) \leqslant \mu(K, \Lambda')$.
- *(Scaling)* For a scalar $\lambda > 0$, $\mu_i(\lambda K, \Lambda) = \frac{1}{\lambda}\mu_i(K, \Lambda)$ since for every $\mu > 0$, $\mu K = (\mu \frac{1}{\lambda})(\lambda K)$.

Notice also that since $\mu = \mu_d$, all of the invariance and monotonicity with respect to inclusion properties also hold for the covering radius.

Remark 2.4. In the definition of the covering minima, it is valid to talk about the minimum of these values instead of their infimum, since K is compact.
Let $L \in \mathcal{A}_{d-i}(\mathbb{R}^d)$ be arbitrary. By the definition of the i-th covering minimum, for every $\varepsilon > 0$, there exist $x_\varepsilon \in K$ and $a_\varepsilon \in \Lambda$ such that $(\mu_i + \varepsilon)x_\varepsilon + a_\varepsilon \in L$. Since Λ is countable, there is an element $a \in \Lambda$ such that there exists a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $a_{\varepsilon_n} = a$ for all $n \in \mathbb{N}$, ie the intersection with L happens in the same lattice translate of K. Since K is compact, $\{x_{\varepsilon_n}\}_{n \in \mathbb{N}}$ has a sequence converging to some $x \in K$. Since L is closed,

$\lim_{n\to\infty}(\mu_i + \varepsilon_n)x_{\varepsilon_n} + a = \mu_i x + a \in L$. Because L was arbitrary, $\mu_i K + \Lambda$ intersects all elements of $\mathcal{A}_{d-i}(\mathbb{R}^d)$. Notice that since $\mu = \mu_d$, this also justifies the use of minimum in the definition of the covering radius of a convex body.

Remark 2.5. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $K + \Lambda$ is a closed set, in the definition of the i-th covering minimum, it is enough to require intersections with all translates of all rational subspaces of dimension $d - i$.

Lemma 2.6. *[14, Lemma 2.3] For a convex body $K \in \mathcal{K}^d$ and lattice $\Lambda \in \mathcal{L}^d$, the following equality holds:*

$$\mu_1(K, \Lambda) = \frac{1}{\omega_\Lambda(K)}.$$

Proof. Let $f \in \Lambda^*$ be a non-zero primitive lattice functional, $\alpha := \max_{x\in K} fx$ and $\beta := \min_{y\in K} fy$. We claim that for any $\mu > 0$, $\mu K + \Lambda$ intersects all hyperplanes of the form $H_\gamma := \{x \in \mathbb{R}^d | fx = \gamma\}$ if and only if $\mu(\alpha - \beta) \geqslant 1$.

Let $a \in \Lambda$ be arbitrary. Then, $fa \in \mathbb{Z}$ by the definition of a dual lattice, and since f is primitive, fa takes all integer values when a passes through Λ. Therefore, for every $p \in a + \mu K$, $fp \in [fa + \mu\beta, fa + \mu\alpha]$. $\mu K + \Lambda$ intersects all hyperplanes H_γ if and only if

$$\bigcup_{a\in\Lambda}[fa + \mu\beta, fa + \mu\alpha] = \bigcup_{n\in\mathbb{Z}}[n + \mu\beta, n + \mu\alpha] = \mathbb{R}.$$

This will happen if and only if $\mu(\alpha - \beta) \geqslant 1$. Since $\omega(K, f) = \alpha - \beta$, this is equivalent to $\mu\omega(K, f) \geqslant 1$. Therefore, $\mu K + \Lambda$ intersects all translates of all rational hyperplanes if and only if for every non-zero primitive $f \in \Lambda^*$, $\mu\omega(K, \Lambda) \geqslant 1$. By the definition of the lattice width and Remark 1.30, this is equivalent to $\mu\omega_\Lambda(K) \geqslant 1$. Since by Remark 2.1 it is enough to check for all hyperplanes with rational directions, the first covering minimum is the minimum of all such μ, we can conclude that $\mu_1(K, \Lambda)\omega_\Lambda(K) = 1$, ie $\mu_1(K, \Lambda) = \frac{1}{\omega_\Lambda(K)}$. ☐

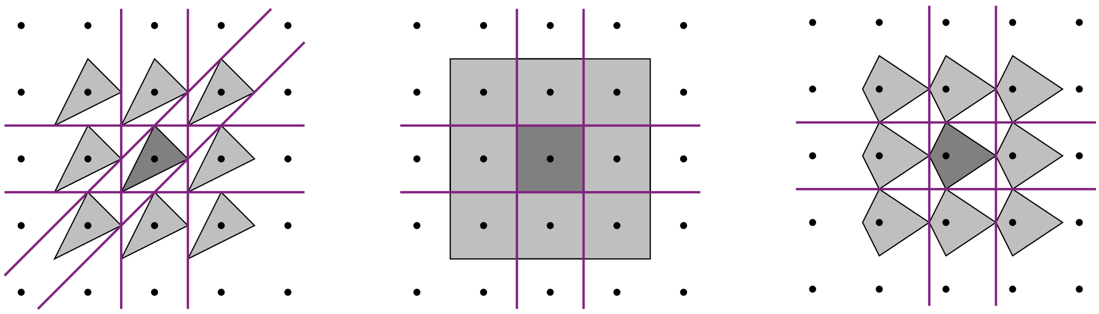
Remark 2.7. Observe that the covering minima are a sequence of functionals connecting the notions of the covering radius and the lattice width, since $\mu_d = \mu$, and Lemma 2.6 connects the first covering minimum with the lattice width by saying they are reciprocal. Thus, the sequence of covering minima interpolates between $\frac{1}{\omega}$ and μ.

Both the lattice width and covering radius have been heavily studied, and specific values are known for many polytopes. Moreover, algorithms for computing both the covering radius and lattice width of a rational polytope in an arbitrary fixed dimension exist. However, calculating any of the covering minima for $2 \leqslant i \leqslant d - 1$ is a much harder task, and they are known for a small family of polytopes, and no algorithms for computation are known.

Motivated by the notion of last covered points for the covering radius of a convex body, we would like to define a similar notion for other covering minima.

Definition 2.8. Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $i \in [d]$. We say that $L \in \mathcal{A}_{d-i}(\mathbb{R}^d)$ a *last covered subspace* if

$$(\mu_i(K,\Lambda)\,\mathrm{int}(K) + \Lambda) \cap L = \varnothing.$$



(a) T_2 (b) C_2 (c) $\mathrm{conv}(-\frac{1}{2}e_1, \frac{3}{2}e_1, -e_2, e_2)$

Figure 2.1: Last Covered Subspaces of dimension 1

Figure 2.1 shows the same three convex bodies as before, scaled by their first covering minima which are all $\frac{1}{2}$, shaded in dark gray, as well as some of their lattice translates in light gray. The red lines are the last covered subspaces of dimension 1 that coincide with the scaling of the original copy of the convex body.

The notion of last covered subspaces was implicitly used by Kannan and Lovász, for example in the following lemma.

Lemma 2.9. *[14, Lem. 2.2] Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $i \in [d]$. Then, there exists $L \in \mathcal{A}_{d-i}(\mathbb{R}^d)$ such that:*

1. *$L \cap (\mu_i(K,\Lambda)\,\mathrm{int}(K) + \Lambda) = \varnothing$ and*
2. *the linear subspace parallel to L is rational.*

Remark 2.10. In other words, this lemma says that last covered subspaces always exist. In a similar vein to Remark 1.46, for any $\mu \geqslant 0$, the following two hold:

1. $\left[\exists L \in \mathcal{A}_{d-i}(\mathbb{R}^d)\ (\mu\,\mathrm{int}(K) + \Lambda) \cap L = \varnothing\right] \Rightarrow \mu \leqslant \mu_i(K,\Lambda)$,

2. $\left[\forall L \in \mathcal{A}_{d-i}(\mathbb{R}^d)\ (\mu \operatorname{int}(K) + \Lambda) \cap L \neq \varnothing\right] \ \Rightarrow\ \mu > \mu_i(K, \Lambda)$.

Kannan and Lovász observe in [14, Remark 1] the following equivalent definition of covering minima, which we prove for completeness.

Proposition 2.11. *Let $K \in \mathcal{K}^d$ be a convex body, $\Lambda \in \mathcal{L}^d$ a lattice and $i \in [d]$. Then*

$$\mu_i(K, \Lambda) = \max\left\{\mu(\pi_L(K), \pi_L(\Lambda)) \mid L \textit{ is a rational } i\textit{-dim subspace of } \mathbb{R}^d\right\}.$$

Proof. ($\geqslant$:) Let $\mu_i := \mu_i(K, \Lambda)$, L be a rational i-dimensional linear subspace of $\mathbb{R}^d$ and $t \in L$ arbitrary. Then, $t + L^\perp \in \mathcal{A}_{d-i}(\mathbb{R}^d)$, therefore $(\mu_i K + \Lambda) \cap (t + L^\perp) \neq \varnothing$. Projecting to L, we get $(\mu_i \pi_L(K) + \pi_L(\Lambda)) \cap \{t\} \neq \varnothing$. Since $t \in L$ was arbitrary, this implies $\mu_i \pi_L(K) + \pi_L(\Lambda) = L$, ie $\mu_i \geqslant \mu(\pi_L(K), \pi_L(\Lambda))$.

($\leqslant$:) By Lemma 2.9, there exists a last covered subspace $L' \in \mathcal{A}_{d-i}(\mathbb{R}^d)$ such that the linear subspace parallel to L' is rational. Let $L' = t + L$, where L is a linear subspace and $t \in L^\perp$. By projecting onto $L^\perp$, which is a rational linear subspace of dimension i, we get $t \notin \mu_i \operatorname{int}(\pi_{L^\perp}(K)) + \pi_{L^\perp}(\Lambda)$, ie $\mu_i \leqslant \mu(\pi_{L^\perp}(K), \pi_{L^\perp}(\Lambda))$. Therefore, since μ_i is less or equal then one of the values on the RHS, it's less or equal then their maximum. □

Remark 2.12. Following this proof, we can also notice a connection between the last covered subspaces and the rational linear subspaces for which the covering radius of the projection will achieve the covering minimum. Sepecifically, if $L' \in \mathcal{A}_{d-i}(\mathbb{R}^d)$ is a last covered subspace, and L the linear subspace parallel to it, then $L^\perp$ is a direction such that $\mu_i(K, \Lambda) = \mu(\pi_{L^\perp}(K), \pi_{L^\perp}\Lambda)$, and additionally, L' projects to a last covered point of $\pi_{L^\perp}(K)$ with respect to the lattice $\pi_{L^\perp}(\Lambda)$.
This also works the other way around – for every projection direction achieving the covering minimum, every subspace parallel to its orthogonal complement that contains a last covered point of the projection will be a last covered subspace of the original convex body.

We would like to point out that this viewpoint on covering minima provides us with lower bounds when looking at a specific convex body, but doesn't suffice for giving global lower bounds that hold for all convex bodies, for example in the spirit of Minkowski's theorems, which are folklore for the covering radius, see for example [10]. Lower bounds of covering minima have been heavily studied, already by Kannan and Lovász ([14]), where they give lower bounds on covering minima which include the lattice point enumerator. Moreover, Merino and Schymura in [9] translate the conjecture of Makai Jr. ([20]) to the language of the first covering minimum, which then resembles Minkowski's First Fundamental Theorem 1.26, and furthermore raise questions of what is the exact bound if we replace the first covering minimum with an arbitrary one, as well as if we study the product of all covering minima with the volume in the vain of Minkowski's Second Fundamental Theorem 1.27. For an overview of inequalities and open questions of this form, we refer to [9], as well as [11] for some similar problems.

2.2 Known Values of all Covering Minima

Since Proposition 2.11 gives a way to see lower bounds for covering minima for a concrete convex body, we can show the first example of calculation of all covering minima for a convex body.

Example 2.13. The covering radius of the standard hypercube is $\mu_i(C_d) = \frac{1}{2}$. This can be seen by noticing that for every $0 < \varepsilon < \frac{1}{2}$, the $(d-i)$-dimensional affine subspace given by $\mathbb{1}_d + \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = \cdots = x_i = 0\}$ does not intersect $(\frac{1}{2} - \varepsilon)C_d + \mathbb{Z}^d$, so by the definition of covering minima, $\mu_i(C_d) \geqslant \frac{1}{2}$. For the other inequality, notice that the projection of C_d to the i-dimensional linear subspace spanned by the first i coordinate axes is C_i, and since $\mu(C_i) = \frac{1}{2}$ (Example 1.41), by Proposition 2.11 we get $\mu_i(C_d) \geqslant \frac{1}{2}$.

The covering minima of unimodular simplices were calculated even in [14], but we will provide a proof in the following proposition that relies on elementary linear algebra for obtaining the upper bound. Kannan and Lovász in [14] used a more involved tool for obtaining the upper bound, involving the successive minima (Lemma 4.1). Merino and Schymura in [9] also commented on how this fact can be seen using the fact that every affine subspace of dimension $d-i$ intersects some i-dimensional coordinate subspace, which is the idea that we have encapsulated in Lemma 6.1.

Proposition 2.14. *Let $S_d \subseteq \mathbb{R}^d$ be the standard d-simplex, ie $S_d := \operatorname{conv}(\mathbb{0}_d, e_1, \dots, e_d)$. Then, for every $i \in [d]$,*

$$\mu_i(S_d) = i.$$

Proof. ($\geqslant$:) If $\pi : \mathbb{R}^d \to \mathbb{R}^i$ is the projection to linear subspace spanned by the first i standard basis vectors, $\pi(S_d) = S_i \times \{\mathbb{0}_{d-i}\}$, and $\pi(\mathbb{Z}^d) = \mathbb{Z}^i \times \{\mathbb{0}_{d-i}\}$. From Example 1.44, $\mu(S_i) = i$, therefore by Proposition 2.11 $\mu_i(S_d) \geqslant i$.

($\leqslant$:) Let $p \in \mathbb{R}^d$ be arbitrary and $v_1, \dots, v_{d-i} \in \mathbb{R}^d$ linearly independent vectors. It would suffice to show that $p + \operatorname{span}_{\mathbb{R}}(v_1, \dots, v_{d-i}) \cap (iS_d + \mathbb{Z}^d) \neq \varnothing$. Since $iS_d = \{x \in \mathbb{R}^d \mid x_j \geqslant 0, \sum_{j=1}^d x_j \leqslant i\}$, it would be enough to find $\lambda_1, \dots, \lambda_{d-i} \in \mathbb{R}$ such that the sum of (positive) fractional parts of the coordinates of $p + \sum\limits_{j=1}^{d-i} \lambda_j v_j$ is at most i.

Since the vectors $v_1, \dots, v_{d-i}$ are linearly independant, there exists a set $I \in \binom{[d]}{d-i}$ of coordinates such that the matrix $[(v_j)_k]_{j \in [d-i], k \in I}$ is an invertible matrix. Choose the coefficients λ such that they satisfy the $(d-i) \times (d-i)$ system of linear equations $\sum_{j=1}^{d-i} \lambda_j (v_j)_k = -p_k$, for all $k \in I$. Therefore, the coordinates corresponding to the indices in the set I of the point $p + \sum\limits_{j=1}^{d-i} \lambda_j v_j$ are 0, therefore have fractional parts 0. There are exactly i coordinates that are not in the set I, and therefore the sum of the fractional parts of the point $p + \sum\limits_{j=1}^{d-i} \lambda_j v_j$ is at most i. □

There aren't many convex bodies for which all covering minima are known.
Merino and Schymura in [9] introduce the following family of polytopes that interpolates between the cube and the crosspolytope, and calculate the values of all their covering minima.

Definition 2.15. Let $d \in \mathbb{N}$, and $i \in [d]$. We define the following d-polytope:

$$P_{d,i} := \operatorname{conv}(\pm e_{j_1} \pm \cdots \pm e_{j_i} \mid 1 \leqslant j_1 < \cdots < j_i \leqslant d) = C_d \cap iC_d^*.$$

Specifically, $P_{d,d} = C_d$ and $P_{d,1} = C_d^*$.

Proposition 2.16. *[9, Prop. 3.3] For every $d \in \mathbb{N}$ and $i \in [d]$, we have:*

$$\mu_j(P_{d,i}) = \begin{cases} \frac{1}{2} & , j \leqslant i \\ \frac{j}{2i} & , j > i. \end{cases}$$

In particular, $\mu_i(C_d) = \frac{1}{2}$ and $\mu_i(C_d^*) = \frac{i}{2}$ for all $i \in [d]$.

The proof of this proposition relies on the fact that the projection of $P_{d,i}$ onto an j-dimensional coordinate subspace L_j is either $P_{j,i}$ or a hypercube, depending on the dimension j and parameter i. Then, they bound the covering minima of $P_{n,i}$ from above by the covering radii of these intersections, which for this family of convex bodies turns out to be sharp.

3 Lattice Polytopes Maximizing the Covering Radius

An upper bound for the covering radius of lattice polytopes is well known, as well as what the maximizers are, as seen in the following proposition.

Proposition 3.1. *If $P \in \mathcal{K}^d$ is a lattice polytope, then:*

$$\mu(P) \leqslant d.$$

Equality holds iff P is a unimodular simplex.

Proof. Let $V \subseteq V(P)$ be any subset of $d+1$ affinely independent vertices of P. Then $S := \operatorname{conv}(V)$ is a lattice simplex contained in P. It suffices to show $\mu(S) \leqslant d$, since $\mu(P) \leqslant \mu(S)$ from Remark 2.3. Since S is a lattice simplex, there exists a linear transformation $A \in \mathbb{Z}^{d\times d}$ of full rank such that $S = AS_d$. Using the properties from Remark 2.3 further, and the fact that $A\mathbb{Z}^d \subseteq \mathbb{Z}^d$, we get:

$$\mu(S, \mathbb{Z}^d) = \mu(AS_d, \mathbb{Z}^d) \leqslant \mu(AS_d, A\mathbb{Z}^d) = \mu(S_d, \mathbb{Z}^d) = d.$$

In Proposition 1.44, we have shown that equality holds for S_d, and therefore for all unimodular simplices since the covering radius is invariant with respect to linear transformations, and the image of $\mathbb{Z}^d$ via a unimodular transformation is $\mathbb{Z}^d$. Now, we only need to show that for any lattice polytope that isn't a unimodular simplex, its covering radius is strictly smaller than d.

Let $S \in \mathcal{K}^d$ be a lattice simplex contained in the starting lattice polytope P. We can assume it has vertices $\mathbb{0}_d, v_1, \dots, v_d$, since the covering radius is translation invariant. Let $\Pi := \left\{ \sum_{i=1}^{d} \alpha_i v_i \mid \alpha_i \in (0,1], \text{ for all } i \in [d] \right\}$ be a half-open parallelepiped. We use the opposite interval than the one in the standard half-open parallelepiped for convenience, but all of the corresponding results from Subsection 1.2 still hold. Since $\Pi + \mathbb{Z}^d = \mathbb{R}^d$ by Proposition 1.12, it is enough to show $\Pi \subseteq d\,\mathrm{int}(P) + \mathbb{Z}^d$, since that would imply $\mathbb{R}^d = \Pi + \mathbb{Z}^d \subseteq d\,\mathrm{int}(P) + \mathbb{Z}^d$, and by Remark 1.46 that would imply $\mu(P) < d$. Now we calculate

$$d\,\mathrm{int}(S) = \left\{ \sum_{i=1}^{d} \alpha_i v_i \mid \alpha_i > 0,\ \sum_{i=1}^{d} \alpha_i < d \right\} \supseteq \Pi \backslash \{v_1 + \dots + v_d\}.$$

By Proposition 1.16, of S isn't unimodular, there exists a lattice point in $a \in \Pi \backslash \{v_1 + \dots + v_d\}$. Then, for a lattice point $b := v_1 + \dots + v_d - a$,

$$v_1 + \dots + v_d \in (\Pi \backslash \{v_1 + \dots + v_d\}) + b \subseteq d\,\mathrm{int}(S) + b.$$

Therefore, $\Pi \subseteq d\,\mathrm{int}(S) + \mathbb{Z}^d \subseteq d\,\mathrm{int}(P) + \mathbb{Z}^d$, ie $\mu(P) < d$.

If S is unimodular, since P is not a unimodular simplex, there exists a vertex v_{d+1} of P such that $v_{d+1} \notin S$. Moreover, we can suppose that v_{d+1} violates the inequality defining the facet opposite of $\mathbb{0}_d$, since in the beginning we made an arbitrary choice of which vertex of S to translate to $\mathbb{0}_d$.
Let $P' := \mathrm{conv}(\mathbb{0}_d, v_1, \dots, v_{d+1}) \subseteq P$. Since $v_1 + \dots + v_d$ is in the relative interior of the facet opposite to the vertex $\mathbb{0}_d$ in the simplex dS, it will be in the interior of the polytope $dP'$by definition of v_{d+1}. Therefore, $\Pi \subseteq d\,\mathrm{int}(P') + \mathbb{Z}^d \subseteq d\,\mathrm{int}(P) + \mathbb{Z}^d$, hence $\mu(P) < d$. □

3.1 Non-Hollow Lattice Polytopes

Notice that all the lattice polytopes that maximize the covering radius are unimodular simplices, and therefore hollow. Codenotti, Santos and Schymura [5] raise the question of what are the upper bound and maximizers for covering radius in the family of all non-hollow lattice polytopes in dimension d.

Since the maximizers in the hollow case are unimodular simplices, a natural candidate for a maximizer in the family of non-hollow lattice polytopes is the terminal simplex, which is the most symmetric lattice simplex with one interior lattice point.

Definition 3.2. For $d \in \mathbb{N}$, the *terminal d-simplex* is the simplex $T_d := \mathrm{conv}(-\mathbb{1}_d, e_1, \ldots, e_d)$. We say that $T \subseteq \mathbb{R}^d$ is a *terminal d-polytope* if T is a lattice d-polytope and can be seen as a direct sum of lattice translates of terminal simplices.

We can see the terminal d-simplex and the d-crosspolytope, which we get as a direct sum of d terminal 1 simplices as the two extremums in the family of terminal d-polytopes, so we can see this as a family of non-hollow lattice polytopes, interpolating between a simplex and a crosspolytope.

Codenotti, Santos and Schymura propose the following conjecture, and prove it in dimensions 2 and 3.

Conjecture 3.3. *[5, Conj. A] Let $P \subseteq \mathbb{R}^d$ be a non-hollow lattice polytope. Then*

$$\mu(P) \leqslant \frac{d}{2},$$

where equality holds iff P is a terminal d-polyope up to a unimodular transformation.

The covering radius was proved to be equal to the value conjectured here, by Merino and Schymura.

Theorem 3.4. *[9, Prop. 4.8.] For every $d \in \mathbb{N}$,*

$$\mu(T_d) = \frac{d}{2}.$$

The proof of this theorem is rather involved. It is based on the fact that the covering radius of the standard simplex $S_d = \mathrm{conv}(\mathbb{0}_d, e_1, \ldots, e_d)$ with respect to a lattice Λ can be seen as the diameter of the directed *quotient lattice graph* (Marklof and Strömbergsson,[21], Lemmas 3 and 4). For more on quotient lattice graphs, we refer to [21].

Additionally, Codenotti, Santos and Schymura ([5], Cor. 2.2) prove that the covering radius is an additive functional with respect to direct sums of convex bodies and lattices. Because terminal d-polytopes are direct sums of translates of terminal simplices, we can conclude that the covering radius of every terminal d-polytope is $\frac{d}{2}$, which verifies correctness of the values of the covering radius in the the conjectured equality cases in Conjecture 3.3.

Regarding the covering minima of terminal simplices, Merino and Schymura conjecture the following:

Conjecture 3.5. *[9, Rem. 4.9] For every $d \in \mathbb{N}$ and $i \in [d]$,*

$$\mu_i(T_d) = \frac{i}{2}.$$

Since projecting T_d to a i-dimensional coordinate subspace gives T_i, Proposition 2.11 for terminal simplices combined with Theorem 3.4 gives us the following lower bound.

Claim 3.6. *Let $d \in \mathbb{N}$ and $i \leqslant d$. Then:*

$$\mu_i(T_d) \geqslant \frac{i}{2}.$$

Proof. Let π be the projection to the space spanned by the first i coordinate vectors. $\pi(T_n) = T_i \times \{\mathbb{0}_{d-i}\}$ and $\pi(\mathbb{Z}^d) = \mathbb{Z}^i \times \mathbb{0}_{d-i}$. From Proposition 2.11 and Theorem 3.4, this implies $\mu_i(T_n) \geqslant \frac{i}{2}$. □

This gives one of the inequalities needed to prove Conjecture 3.3. The other inequality needed would be of the form of an upper bound of the covering minima of a convex body. The definition of covering minima is technically a tool for obtaining upper bounds, but checking if a subset of $\mathbb{R}^d$ intersects all affine subspaces of a fixed dimension is rather difficult.

The upper bound for the covering radius of an arbitrary non-hollow lattice polytope and the values of the covering minima of terminal simplices were connected by Codenotti, Santos and Schymura in the following theorem.

Theorem 3.7. *[5, Thm. 1.2] For every $d \in \mathbb{N}$, the following are equivalent:*

i) $\mu(P) \leqslant \frac{i}{2}$ *for every $i \leqslant d$ and every non-hollow lattice i-polytope P.*

ii) $\mu_i(T_n) = \frac{i}{2}$ *for every $n \geqslant d$ and every $i \leqslant d$.*

The implication $i) \Rightarrow ii)$ follows from Claim 3.6 and the fact that the i-th covering minimum of T_n is the maximum of covering radii of rational i-projections of T_n, which are all non-hollow lattice polytopes since T_n is a non-hollow lattice polytope. The implication $ii) \Rightarrow i)$ is shown by for a given non-hollow lattice i-polytope P, finding an $n \in \mathbb{N}$ and a rational projection $\pi : \mathbb{R}^n \to \mathbb{R}^i$ such that $\pi(T_n) = P$. Then, since P can be seen as a rational projection of T_n, $\frac{i}{2} = \mu_i(T_n) \geqslant \mu(P, \pi(\mathbb{Z}^n)) \geqslant \mu(P)$.

Notice that in Theorem 5.1 *ii)* the value of the i-th covering minimum is required to be $\frac{i}{2}$ for *all* terminal simplices, not only for the ones in dimensions up to d, so simple inductive arguments for proving that *ii)* holds do not suffice.

Conjecture 3.3 is proven to hold in dimensions 2 and 3 [5, Cor. 3.6, Thm. 3.13]. More specifically, they have proven that every non-hollow lattice polygon has a covering radius at most one, and every non-hollow lattice 3-polytope has covering radius at most $\frac{3}{2}$. These proofs use the classification of inclusion minimal non-hollow lattice polytopes, which is folklore in dimension 2, and is done in [15, Thm. 3.1] for dimension 3. This approach requires both the classification of such objects, which is not known in higher dimensions, and furthermore bounding their covering radii.

From the fact that Conjecture 3.3 is proved in dimensions 2 and 3, and Theorem 5.1, we can conclude that for every $n \in \mathbb{N}$, $\mu_1(T_n) = \frac{1}{2}$, $\mu_2(T_n) = 1$ and $\mu_3(T_n) = \frac{3}{2}$.

Trying to tackle this problem from the side of Conjecture 3.5 requires understanding the behaviour of more dificult to grasp functionals (the covering minima) of very specific polytopes – the terminal simplices. Since the lower bound is already known, as in Claim 3.6, one of the logical next steps would be to investigate possible upper bounds on covering minima.

4 Known Upper Bounds for Covering Minima of Convex Bodies

Kannan and Lovász give the following upper bound on covering minima, involving the previous covering minimum and an appropriate successive minimum of the difference body. We will present the proof of this bound, due to Kannan and Lovász, to emphasise the fact that last covered subspaces can be used as a tool for translating results on covering radii to results on covering minima.

Lemma 4.1. *[14, Lem. 2.5] For a convex body $K \in \mathcal{K}^d$, lattice $\Lambda \in \mathcal{L}^d$ and $i \in [d]$, the following inequality holds:*

$$\mu_{i+1}(K, \Lambda) \leqslant \mu_i(K, \Lambda) + \lambda_{d-i}(K - K, \Lambda).$$

Proof. Denote by $\mu_i := \mu_i(K, \Lambda)$ and by $\lambda_i := \lambda_i(K - K, \Lambda)$ for all $i \in [d]$.
First, let's prove this claim for $i = d - 1$. Let $v \in \lambda_1 \cdot (K - K)$ be a non-zero lattice vector. Since all covering minima and successive minima of the difference body are translatory invariant, we can translate K so that $\mathbb{0}_d, v \in \lambda_1 K$. Let $p \in \mathbb{R}^d$ be arbitrary. By the definition of the $(d-1)$-st covering minimum, $\mu_{d-1}K + \Lambda$ has to intersect the line $p + \mathrm{span}_{\mathbb{R}}(v)$. Therefore, there exist $x \in K$, $a \in \Lambda$ and $t \in \mathbb{R}$ such that $p - tv = \mu_{d-1}x + a$. Then, $p = \mu_{d-1}x + (t - \lfloor t \rfloor)v + \lfloor t \rfloor v + a$. Since $v \in \lambda_1 K$, and $\mathbb{0}_d \in K$, from $t - \lfloor t \rfloor \in [0, 1)$ we can conclude $(t - \lfloor t \rfloor)v \in \lambda_1 K$. Moreover, since $\mathbb{0}_d \in K$ and K is convex,

$\mu_{d-1}x + (t - \lfloor t \rfloor)v \in (\mu_d + \lambda_1)K$. Since $a, \lfloor t \rfloor v \in \Lambda$, this means that $p \in (\mu_{d-1} + \lambda_1)K + \Lambda$, ie $\mu_d \leqslant \mu_{d-1} + \lambda_1$.

For $i \in [d-2]$, we will make use of the notion of last covered subspaces. By Lemma 2.9, there exists a last covered subspace $L \in \mathcal{A}_{d-i-1}(\mathbb{R}^d)$. Denote by $K' := \pi_{L^\perp}(K)$, $\Lambda' := \pi_{L^\perp}(\Lambda)$, and by μ', μ'_j, λ'_j the corresponding covering radius, minima and successive minima for $j \in [i+1]$. By Remark 2.12, $\mu_{i+1} = \mu'$. From the first part of this proof applied to the convex body K' and lattice Λ', we can conclude $\mu' \leqslant \mu'_i + \lambda'_1$. Since the i-th covering minimum is the maximum of covering radii over all rational projections of dimension i, and the set of all rational i-dimensional projections of $\mathbb{R}^d$ with respect to Λ is a superset of all those of $\pi_{L^\perp}(\mathbb{R}^d)$ with respect to Λ', we see that $\mu'_i \leqslant \mu_i$. Let $v_1, \ldots, v_{d-i} \in \lambda_{d-i}(K - K)$ be linearly independent lattice vectors. Then, since $\dim(\mathrm{Ker}(\pi_{L^\perp})) = d - i - 1$, there has to be at least one non-zero vector among $\pi_{L^\perp}(v_1), \ldots, \pi_{L^\perp}(v_{d-i}) \in \lambda_{d-i}(K' - K')$. As all of them are lattice vectors, this implies $\lambda'_1 \leqslant \lambda_{d-i}$. Therefore, $\mu_{i+1} = \mu' \leqslant \mu'_i + \lambda'_1 \leqslant \mu_i + \lambda_{d-i}$. □

In [11], Henk, Schymura and Xue notice that in this proof, we can replace the successive minima with yet another functional, *packing minima*, and the same proof suffices. We refer to [11] for a review on packing minima.

Example 4.2. This upper bound will be sufficient for calculating all the covering minima of the d-crosspolytope. Namely, since the projection of C_d^* to the first i coordinates is C_i^*, we know $\mu_i(C_d^*) \geqslant \mu(C_i^*) = \frac{i}{2}$ (Example 1.42). From Example 1.24, we know for all $j \in [d]$ that $\lambda_j(C_d^*) = 1$, and therefore $\lambda_j(C_d^* - C_d^*) = \lambda_j(2C_d^*) = \frac{1}{2}$. Since $\mu_1(C_d^*) = \frac{1}{\omega(C_d^*)} = \frac{1}{2}$ by Example 1.39, successive application of Theorem 4.1 gives us $\mu_i(C_d^*) \leqslant \mu_1(C_d^*) + (i-1)\frac{1}{2} = \frac{i}{2}$.

Since successive minima as well as difference bodies are also difficult to calculate, the next bound given by Kannan and Lovász modifies the previous one to involve the first covering minimum, ie the reciprocal of the lattice width, instead of the succesive minima of difference bodies.

Lemma 4.3. *[14, Lem. 2.6] For a convex body $K \in \mathcal{K}^d$, lattice $\Lambda \in \mathcal{L}^d$ and every $1 \leqslant i \leqslant d-1$,*

$$\mu_{i+1}(K, \Lambda) \leqslant \mu_i(K, \Lambda) + c(i+1)\mu_1(K, \Lambda),$$

where c is an absolute constant.

The proof of this lemma for $i = d-1$ follows by plugging $\lambda_1(K-K, \Lambda)\lambda_1((K-K)^*, \Lambda^*) \leqslant cd$, and $\mu_1(K, \Lambda) = \lambda_1((K-K)^*, \Lambda^*)$ into the previous lemma. This inequality is a consequence of Minkowski's First Fundamental theorem 1.26 and the fact that there exist absolute constants c_1, c_2 such that for every d-dimensional o-symmetric convex body, $\frac{c_1^d}{d^d} \leqslant \mathrm{vol}(S)\,\mathrm{vol}(S^*) \leqslant \frac{c_2^d}{d^d}$. The lower bound, which is the one we use, is due to Bourgain and Milman ([2]), and it was conjectured by Mahler ([19]) that $c_1 = 4$, with equality holding for the hypercube, but also for the crosspolytope and a whole family interpolating between the two. The upper bound is due to Santaló ([24]), and equality holds for a ball. The absolute constant c in the statement of this lemma is actually $\frac{4}{c_1}$, which means that $c \geqslant 1$. For the cases $i \in [d-2]$, it is again sufficient to reduce to the first case by looking at last covered subspaces.

The following theorem is Kannan and Lovászes main bound on covering minima, and comes directly from sucessively applying the previous lemma.

Theorem 4.4. *[14, Thm. 2.7] For a convex body $K \in \mathcal{K}^d$, lattice $\Lambda \in \mathcal{L}^d$ and every $1 \leqslant i \leqslant d-1$,*

$$\mu_i(K,\Lambda) \leqslant c\binom{i}{2}\mu_1(K,\Lambda),$$

where c is an absolute constant.

When restricting to o-symmetric convex bodies, the following result is proven.

Theorem 4.5. *[14, Thm. 2.13] For a o-symmetric convex body $K \in \mathcal{K}^d$, lattice $\Lambda \in \mathcal{L}^d$ and every $1 \leqslant i \leqslant d-1$,*

$$\mu_{i+1}(K,\Lambda) \leqslant 2\mu_i(K,\Lambda).$$

It is again sufficient to prove this for $i = d-1$ and reduce the other cases to that one by observing last covered subspaces. The proof utilizes the norm defined by the o-symmetric body, and the methods used cannot be generalized to general convex bodies.

5 Upper Bounds via Projections

One of the tools introduced in [5] gives upper bounds on the covering radius of a convex body, given in the next lemma.

Lemma 5.1. *[5, Lem. 2.1] Let $K \in \mathcal{K}^d$ be a convex body containing the origin, and let $\pi : \mathbb{R}^d \to \mathbb{R}^l$ be a linear projection to a rational l-subspace. Let $Q = K \cap \pi^{-1}(0)$ and let $L = \pi^{-1}(0)$ be the linear subspace spanned by Q. Then, we have*

$$\mu(K,\mathbb{Z}^d) \leqslant \mu(Q,\mathbb{Z}^d \cap L) + \mu(\pi(K),\pi(\mathbb{Z}^d)).$$

We generalize this to a similar upper bound on the covering minima of a convex body via the covering minima of its projections and intersections with rational subspaces.

Notation 5.2. For $K \in \mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$, for notational convenience we additionally define $\mu_0(K,\Lambda) := 0$. Notice that this definition agrees with the definition of covering minima, because indeed, $0K + \Lambda = \{\mathbb{0}_d\} + \Lambda = \Lambda$ intersects $\mathbb{R}^d$, which is the only d-dimensional affine subspace od $\mathbb{R}^d$.

Theorem 5.3. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$ and $V \subseteq \mathbb{R}^d$ a rational linear subspace of dimension l and $i \in [d]$. If by π_V we denote the natural projection of $\mathbb{R}^d$ to V, the following holds:*

$$\mu_i(K,\Lambda) \leqslant \max_{\substack{0\leqslant j\leqslant l\\ 0\leqslant i-j\leqslant d-l}} \mu_j(\pi_V(K),\pi_V(\Lambda)) + \mu_{i-j}(K\cap V^\perp, \Lambda\cap V^\perp)$$

Proof. By definition, $\mu_i(K,\Lambda)$ is minimal such that $\mu_i(K,\Lambda)K+\Lambda$ intersects every $(d-i)$-dimensional affine subspace of $\mathbb{R}^d$.
Let $x+y+U$ be an arbitrary $(d-i)$-dimensional affine subspace of $\mathbb{R}^d$, where $x \in V$, $y \in V^\perp$, $U \leqslant \mathbb{R}^d$ linear subspace.
Let $U_V = \pi_V(U)$ and $U_{V^\perp} = U \cap V^\perp$. Notice that if we look at the restriction $\pi_V|_U$, U_V is the image of this map, and $U_{V^\perp}$ it's kernel, therefore $\dim(U) = \dim(U_V) + \dim(U_{V^\perp})$.
Let $\dim(U_V) = l-j$, $\dim(U_{V^\perp}) = (d-i)-(l-j) = (d-l)-(i-j)$. For brievity, let $\mu_j := \mu_j(\pi_V(K),\pi_V(\Lambda))$ and $\mu_{i-j} := \mu_{i-j}(K\cap V^\perp,\Lambda\cap V^\perp)$.
Since $x+U_V = x+\pi_V(U) = \pi_V(x+U)$ is a $(l-j)$-dim affine subspace of V, there exist $u_V \in U$, $p \in K$, $a \in \Lambda$ such that

$$x+\pi_V(u_1) = \mu_j\pi_V(p)+\pi_V(a)$$

Let $y_{u_1} = u_1-\pi_V(u_1)$, $y_p = p-\pi_V(p)$ and $y_a = a-\pi_V(a)$. Notice that $y_{u_1}, y_p, y_a \in V^\perp$, and because $V^\perp$ is a linear space, $y' := y+y_{u_1}-\mu_j y_p - y_a \in V^\perp$. Since $(y'+U)\cap V^\perp = y'+(U\cap(-y'+V^\perp)) = y'+(U\cap V^\perp)$ is a $((d-l)-(i-j))$-dim affine subspace of $V^\perp$, there exist $u_2 \in U\cap V^\perp$, $q \in K$ and $b \in \Lambda\cap V^\perp$ such that

$$y'+u_2 = \mu_{i-j}q+b$$
$$\Rightarrow y+y_{u_1}-\mu_j y_p - y_a + u_2 = \mu_{i-j}q+b$$

Now, adding up the two equalities we get:

$$x+\pi_V(u_1)+y+y_{u_1}-\mu_j y_p - y_a+u_2 = \mu_j\pi_V(p)+\pi_V(a)+\mu_{i-j}q+b$$

$$\Rightarrow x+y+\pi_V(u_1)+y_{u_1}+u_2 = \mu_j(\pi_V(p)+y_p)+\pi_V(a)+uy_a+\mu_{i-j}q+b =$$

$$\Rightarrow x+y+u_1+u_2 = \mu_j p + a + \mu_{i-j}q+b$$

$$\Rightarrow x+y+u_1+u_2 = (\mu_j+\mu_{i-j})\left(\frac{\mu_j}{\mu_j+\mu_{i-j}}p+\frac{\mu_{i-j}}{\mu j+\mu_{i-j}}q\right)+a+b.$$

Here, LHS is in $x+y+U$, and RHS is in $(\mu_j+\mu_{i-j})K+\Lambda$. Since $x+y+U$ was arbitrary, $\max\{\mu_j(\pi_V(K),\pi_V(\Lambda))+\mu_{i-j}(K\cap V^\perp,\Lambda\cap V^\perp) \mid 0\leqslant j\leqslant l, 0\leqslant i-j\leqslant d-l\}K$ intersects every $(d-i)$-dim subspace of $\mathbb{R}^d$, therefore $\mu_i(K,\Lambda) \leqslant \max\{\mu_j(\pi_V(K),\pi_V(\Lambda))+\mu_{i-j}(K\cap V^\perp,\Lambda\cap V^\perp) \mid 0\leqslant j\leqslant l, 0\leqslant i-j\leqslant d-l\}$. □

5.1 Covering Minima of Direct Sums

Motivated by the conjecture that the terminal d-polytopes are the family of maximizers for the covering radius in the family of non-hollow lattice polytopes, and the connection of this conjecture to specific values of the covering minima, we want to investigate how these functionals behave with respect to direct sums.

It is known that the covering radius is an additive functional with respect to direct sum, as can be seen in [5, Cor. 2.2].

On the other hand, the lattice width is not an additive functional with respect to direct sums. The lattice width of the direct sum is the minimum of the lattice widths of the summands [6, Thm. 2.2]. We connect these two results, and give the answer to the question of how covering minima interact with direct sums in the following theorem.

Theorem 5.4. *Let $\mathbb{R}^d = V \oplus W$, $\dim(V) = l$, $\dim(W) = d - l$, $K \subseteq V$ and $L \subseteq W$ convex bodies that contain the origin, $\Lambda \subseteq V$ and $\Gamma \subseteq W$ lattices, and $i \in [d]$. Then:*

$$\mu_i(K \oplus L, \Lambda \oplus \Gamma) = \max_{\substack{0 \leqslant j \leqslant l \\ 0 \leqslant i-j \leqslant d-l}} \mu_j(K, \Lambda) + \mu_{i-j}(L, \Gamma).$$

Proof. ($\leqslant$:) By using Theorem 5.3 to the subspace V, we get the statement as $V^\perp = W$ and by the definition of direct sum, $\pi_V(K \oplus L) = K$, $\pi_V(\Lambda \oplus \Gamma) = \Lambda$, $(K \oplus L) \cap W = L$ and $(\Lambda \oplus \Gamma) \cap W = \Gamma$.

($\geqslant$:) For all j such that $0 \leqslant j \leqslant l$ and $0 \leqslant i-j \leqslant d-l$, we want to show $\mu_i(K \oplus L, \Lambda \oplus \Gamma) \geqslant \mu_j(K, \Lambda) + \mu_{i-j}(L, \Gamma)$.

First, from the definitions of direct sums of convex bodies and lattices, we notice that every projection π to a i-dim rational subspace of V has $\pi(K \oplus L) = \pi(K)$ and $\pi(\Lambda \oplus \Gamma) = \pi(\Lambda)$, therefore $\mu_i(K \oplus L, \Lambda \oplus \Gamma) \geqslant \mu_i(K, \Lambda)$ (if $i \leqslant l$, ie if $\mu_i(K, \Lambda)$ makes sense, then). Similarly, when it makes sense, $\mu_i(K \oplus L, \Lambda \oplus \Gamma) \geqslant \mu_i(L, \Gamma)$.

Now, we can assume $j, i - j \neq 0$, or moreover, $\mu_j := \mu_j(K, \Lambda) > 0$ and $\mu_{i-j} := \mu_{i-j}(L, \Gamma) > 0$.

Suppose the countrary, and take $0 < c < \mu_j$ and $0 < c' < \mu_{i-j}$ such that $c + c' = \mu_i(K \oplus L, \Lambda \oplus \Gamma)$. Then there exists a $(l - j)$-dimensional linear subspace $U_V \leqslant V$ and $x \in V$ such that $(x + U_V) \cap cK = \varnothing$; similarly, there exists a $(d - l - i + j)$-dim linear subspace $U_W \leqslant W$ and $y \in W$ such that $(y + U_W) \cap c'L = \varnothing$.

Since $\mu_i(K \oplus L, \Lambda \oplus \Gamma) = c + c'$ and $x + y + (U_V \oplus U_W)$ is a $(d - i)$-dim affine subspace of $\mathbb{R}^d$, it intersects $(c + c')(K \oplus L) + (\Lambda \oplus \Gamma)$, ie there exist $v \in U_V$, $w \in U_W$, $p \in K$, $q \in L$, $\lambda \in [0, 1]$,$a \in \Lambda$ and $b \in \Gamma$ s.t:

$$x + y + v + w = (c + c')(\lambda p + (1 - \lambda)q) + a + b.$$

Since the sum is direct, this implies $x + v = (c + c')\lambda p + a$ and $y + w = (c + c')(1 - \lambda)q + b$. Because $\mathbb{0}_d \in K$ and $\mathbb{0}_d \in L$, which are convex sets, $\lambda p \in K$ and $(1 - \lambda)q \in L$. Moreover,

since $(x + U_V) \cap cK = \emptyset$ and $(y + U_W) \cap c'L = \emptyset$, we can conclude $(c + c')\lambda > c$ and $(c + c')(1 - \lambda) > c'$. This is equivalent to $c'\lambda > c(1 - \lambda)$ and $c(1 - \lambda) > c'\lambda$, which is a contradiction.
Therefore, we showed $\mu_i(K \oplus L, \Lambda \oplus \Gamma) \geqslant \mu_j(K, \Lambda) + \mu_{i-j}(L, \Gamma)$ for every relevant j, ie $\mu_i(K \oplus L, \Lambda \oplus \Gamma) \geqslant \max\{\mu_j(K) + \mu_{i-j}(L) | 0 \leqslant j \leqslant l, 0 \leqslant i - j \leqslant d - l\}$.

□

Specifically, this is a generalization of the two forementioned results:

- *Covering radius of direct sums:*

$$\mu(K \oplus L, \Lambda \oplus \Gamma) = \mu_d(K \oplus L, \Lambda \oplus \Gamma) = \max_{\substack{0 \leqslant j \leqslant l \\ 0 \leqslant d-j \leqslant d-l}} \mu_j(K, \Lambda) + \mu_{d-j}(L, \Gamma) =$$

$$= \mu_l(K, \Lambda) + \mu_{d-l}(L, \Gamma) = \mu(K, \Lambda) + \mu(L, \Gamma).$$

- *Lattice width of direct sums:*

$$\omega_{\Lambda \oplus \Gamma}(K \oplus L) = \frac{1}{\mu_1(K \oplus L, \Lambda \oplus \Gamma)} = \frac{1}{\max\limits_{\substack{0 \leqslant j \leqslant l \\ 0 \leqslant 1-j \leqslant d-l}} \mu_j(K, \Lambda) + \mu_{d-j}(L, \Gamma)} =$$

$$= \frac{1}{\max\{\mu_1(K, \Lambda), \mu_1(L, \Gamma)\}} = \min\left\{\frac{1}{\mu_1(K, \Lambda)}, \frac{1}{\mu_1(L, \Gamma)}\right\} = \min\{\omega_\Lambda(K), \omega_\Gamma(L)\}.$$

It is interesting that the behaviour of the operator of direct sum for the covering radius and lattice width can be unified and proved in the same way, going through covering minima. This raises a question wheter there are more results either on the covering radius or on the lattice width side that could be modified to work for all covering minima.

Now that we know how all covering minima behave with direct sums, we can see that knowing all the covering minima of all terminal simplices would also result in knowing all covering minima of terminal polytopes. More specifically:

Corollary 5.5. *Let T be a terminal d-polytope. If we assume that Conjecture 3.5 holds in all dimensions up to d, ie $\mu_j(T_k) = \frac{j}{2}$, for all $j \leqslant k \leqslant d$, then for all $i \in [d]$,*

$$\mu_i(T) = \frac{i}{2}.$$

Proof. Let $k \in \mathbb{N}$, $l_1, \ldots, l_k \in \mathbb{N}$ s. t. $\sum_{i=1}^{k} l_i = d$ and for $j \in [k]$, $u_j \in \mathbb{R}^{k_j}$ such that $\mathbb{0}_{l_j} \in u_j + S(\mathbb{1}_{l_j+1})$. An arbitrary terminal d-polytope can be seen as $T = (u_1 + S(\mathbb{1}_{l_1+1})) \oplus \cdots \oplus (u_k + S(\mathbb{1}_{l_k+1}))$, and this decomposition into a direct sum agrees with the decomposition of $\mathbb{R}^d$ into $\mathbb{R}^d = \mathbb{R}^{l_1} \oplus \cdots \oplus \mathbb{R}^{l_k}$.
Since we assume that Conjecture 3.5 holds in dimensions up to d, we know that for every $j \in [k]$ and every $0 \leqslant s \leqslant l_j$, $\mu_s(u_j + S(\mathbb{1}_{l_j+1})) = \frac{s}{2}$.
For any $i \in [d]$, using this and applying Theorem 5.4 we get:

$$\mu_i(T) = \max\left\{ \sum_{j=1}^{k} \frac{s_j}{2} \,\middle|\, 0 \leqslant s_j \leqslant l_j,\ \sum_{j=1}^{k} s_j = i \right\}$$
$$\Rightarrow \mu_i(T) = \frac{i}{2}.$$

□

Going back to the two equivalent conjectures regarding the upper bound for the covering radius of a non-hollow lattice polytope, and values of covering minima of the terminal simplices, using this corollary we get a stronger conjecture, and the associated stronger version of Theorem as follows.

Conjecture 5.6. *For every $d \in \mathbb{N}$, every terminal d-polytope T and every $i \in [d]$,*

$$\mu_i(T) = \frac{i}{2}.$$

Theorem 5.7. *For every $d \in \mathbb{N}$, the following are equivalent:*

i) $\mu(P) \leqslant \frac{i}{2}$ *for every $i \leqslant d$ and every non-hollow lattice i-polytope P.*

ii) $\mu_i(T_n) = \frac{i}{2}$ *for every $n \geqslant d$ and every $i \leqslant d$.*

iii) $\mu_i(T) = \frac{i}{2}$ *for every $n \geqslant d$, every terminal n-polytope T and every $i \leqslant d$.*

5.2 Terminal Simplices

Our next goal is to try to utilize the bound we give in Theorem 5.3 to get an upper bound for the covering minima of terminal simplices. First, notice that the bound from this theorem depends on covering minima of the projection to a linear subspace, as well as those of intersections with the orthogonal complement of said linear subspace. One could notice that in this case, it is convenient to work with coordinate subspaces, since the projection of a terminal simplex to a coordinate subspace is the terminal simplex of the appropriate dimension. Now, we would like to investigate what is the intersection of a terminal simplex and a given coordinate subspace.

Notation 5.8. For convenience, while utilizing Theorem 5.3, we will just write $\max_k$ instead of actually writing what values k takes – it will always be assumed that k takes values such that all the covering minima in the expression make sense, including 0.

Codenotti, Santos and Schymura in [5] introduce the following family of weighted versions of terminal simplices which will be of much use for our purposes.

Definition 5.9. Let $\omega = (\omega_0, \ldots, \omega_d) \in \mathbb{R}^{d+1}_{>0}$ be a vector of weights. We define the following family of simplices:

$$S(\omega) := \operatorname{conv}(\omega_0 \cdot (-\mathbb{1}), \omega_1 \cdot e_1, \ldots, \omega_d \cdot e_d) \in \mathcal{K}^d.$$

Specifically, $S(\mathbb{1}_{d+1}) = T_d$.

In the following lemma, we describe the convex body that one gets when intersecting a terminal simplex with an arbitrary coordinate subspace.

Lemma 5.10. *Let $i \in [d]$ and let L be a coordinate subspace of $\mathbb{R}^d$ of dimension i. Then, $T_d \cap L = S((\frac{1}{d-i+1}, 1, \ldots, 1))$, where the weight vector has $i+1$ entries, and equality means really the equality if we restrict to the coordinates contained in L.*

Proof. Since T_d is symmetric with respect to the coordinate directions, we can assume $L = \mathbb{R}^i \times \{\mathbb{0}_{d-i}\}$. Then, this statement is equivalent to showing $T_d \cap (\mathbb{R}^i \times \{\mathbb{0}_{d-i}\}) = S((\frac{1}{d-i+1}, 1, \ldots, 1)) \times \{\mathbb{0}_{d-i}\} = \operatorname{conv}(\frac{1}{d-i+1}(-\mathbb{1}_i), e_1, \ldots, e_i)$.
($\supseteq$:) Since $e_1, \ldots, e_i$ are contained in $T_d = \operatorname{conv}(-\mathbb{1}_d, e_1, \ldots, e_d)$ and $\mathbb{R}^i \times \{\mathbb{0}_{d-i}\}$, and

$$\frac{1}{d-i+1}(-\mathbb{1}_i) = \frac{1}{d-i+1}(-\mathbb{1}_d) + \sum_{j=i+1}^{d} \frac{1}{d-i+1} e_j$$

is also contained in both T_d and $\mathbb{R}^i \times \{\mathbb{0}_{d-i}\}$, their convex hull is contained in $T_d \cap (\mathbb{R}^i \times \{\mathbb{0}_{d-i}\})$.
($\subseteq$:) Let $x = \alpha_0(-\mathbb{1}_d) + \sum_{j=1}^{d} \alpha_j e_j$ be an arbitrary element in $T_d \cap (\mathbb{R}^i \times \{\mathbb{0}_{d-i}\})$, where $\alpha_j \geqslant 0$ and $\sum_{j=0}^{d} \alpha_j = 1$. Then for all $i+1 \leqslant j \leqslant d$, $\alpha_j = \alpha_0$ because the last $d-i+1$ coordinates have to be 0. Rewriting, we get:

$$x = \alpha_0 \left(-\mathbb{1}_d + \sum_{j=i+1}^{d} e_j \right) + \sum_{j=1}^{i} \alpha_j e_j = (d-i+1)\alpha_0 \left(\frac{1}{d-i+1}(-\mathbb{1}_i) \right) + \sum_{j=1}^{i} \alpha_j e_j$$

Since $(d-i+1)\alpha_0 + \sum_{j=1}^{i} \alpha_j = \sum_{j=0}^{d} \alpha_j = 1$, it follows that $x \in \operatorname{conv}(\frac{1}{d-i+1}(-\mathbb{1}_i), e_1, \ldots, e_i)$. $\square$

From this lemma, we can see that for explicitly utilizing our projection bound to terminal simplices, we would have to know all the values of covering minima of weighted versions of terminal simplices. In [5], Codenotti, Santos and Schymura propose the following.

Conjecture 5.11. *[5, Conj. 5.3] For every $\omega \in \mathbb{R}^{d+1}_{>0}$ with $\omega_0 \leqslant \cdots \leqslant \omega_d$, and every $i \in [d]$, the i-th covering minimum of $S(\omega)$ is attained by the projection to the first i coordinates, ie:*

$$\mu_i(S(\omega)) = \frac{\sum\limits_{0 \leqslant j < k \leqslant i} \frac{1}{\omega_j \omega_k}}{\sum\limits_{j=0}^{i} \frac{1}{\omega_j}}.$$

Of course, assuming that this conjecture is true would include assuming we know all covering minima of terminal simplices, since $S(\mathbb{1}_{d+1}) = T_d$. Nevertheless, this Conjecture is known to hold for $i = d$ (Theorem 6.4) and $i = 1$, because the covering radius of line segments is known and therefore is less difficult to check manually what the first covering minimum of a convex body is. Since the case $i = d$ is of no use for our projection bound, we will utilize the case where the intersection is of dimension 1, ie projection is to a $d-1$ dimensional coordinate subspace.

Corollary 5.12. *For every $d \in \mathbb{N}$ and $2 \leqslant i \leqslant d$,*

$$\mu_i(T_d) \leqslant \frac{1}{2} + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1}.$$

Proof. Observe the coordinate hyperplane $L = \mathbb{R}^{d-1} \times \{0\}$. Then, $\pi_L(T_d) = T_{d-1} \times \{0\}$, and as seen in Lemma 5.10, $T_d \cap L^\perp = \{\mathbb{0}_{d-1}\} \times S(\frac{1}{d}, 1) = \{\mathbb{0}_{d-1}\} \times \operatorname{conv}(-\frac{1}{d}e_d, e_d)$. Since the ambient space does not matter for covering minima purposes, and $\pi_L(\mathbb{Z}^d) = \mathbb{Z}^{d-1} \times \{0\}$ and $\mathbb{Z}^d \cap L^\perp = \{\mathbb{0}_{d-1}\} \times \mathbb{Z}$, we can just see these as T_{d-1} and $[-\frac{1}{d}, 1]$ in corresponding standard lattices. Notice also that $\mu([-\frac{1}{d}, 1]) = \frac{d}{d+1}$, since the covering radius of every segment is just the scaling needed to get its lenght to be 1. Then, Theorem 5.3 applied to T_d and L gives the following:

$$\mu_i(T_d) \leqslant \max_k \mu_k(T_{d-1}) + \mu_{i-k}([-\frac{1}{d}, 1]) = \max\left\{\mu_i(T_{d-1}), \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\right\}.$$

Notice that on the right hand side, the dimension of the terminal simplices observed has dropped. By successive application of this inequality, we can get to one of the values that we know – the first covering minimum of a terminal simplex, or the covering radius of a terminal simplex. Specifically, by applying this inequality to the first element in the set we're taking the maximum of, we get:

$$\mu_i(T_d) \leqslant \max\left\{\mu_i(T_{d-2}), \mu_{i-1}(T_{d-2}) + \frac{d-1}{d}, \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\right\}.$$

Let's first prove $\mu_i(T_d) \leqslant \max\{\frac{i}{2}, \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\}$.
Since T_{d-1} projects to T_{d-2} when projecting out the last coordinate, $\mu_{i-1}(T_{d-2}) \leqslant \mu_{i-1}(T_{d-1})$. Additionally, $\frac{d-1}{d} < \frac{d}{d+1}$, therefore $\mu_{i-1}(T_{d-2}) + \frac{d-1}{d} < \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}$, and the former can be removed from the set we are maximizing over. This brings us to $\mu_i(T_d) \leqslant \max\{\mu_i(T_{d-2}), \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\}$, ie we just dropped the dimension of the terminal simplex by 1 again in the first element of the set we're maximizing. Repeating this process $d-i$ times in total, we get to the covering radius of a terminal simplex, which is a value we know:

$$\mu_i(T_d) \leqslant \max\left\{\mu_i(T_i), \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\right\} = \max\left\{\frac{i}{2}, \mu_{i-1}(T_{d-1}) + \frac{d}{d+1}\right\}.$$

Applying this inequality to the term $\mu_{i-1}(T_{d-1})$ on the right hand side, we get:

$$\mu_i(T_d) \leqslant \max\left\{\frac{i}{2}, \frac{i-1}{2} + \frac{d}{d+1}, \mu_{i-2}(T_{d-2}) + \frac{d-1}{d} + \frac{d}{d+1}\right\}.$$

Now, notice that since $d \geqslant 1$, $\frac{d}{d+1} \geqslant \frac{1}{2}$, ie $\frac{i-1}{2} + \frac{d}{d+1} \geqslant \frac{i}{2}$. Moreover, for every $0 \leqslant j \leqslant i$, since $d > i$, $\frac{d-j}{d-j+1} \geqslant \frac{1}{2}$. Therefore, applying the inequality $\mu_{i-j}(T_{d-j}) \leqslant \max\{\frac{i-j}{2}, \mu_{i-j-1}(T_{d-j-1}) + \frac{d-j}{d-j+1}\}$ successively $i-1$ times for $0 \leqslant j \leqslant i-2$ brings us to:

$$\mu_i(T_d) \leqslant \max\left\{\frac{1}{2} + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1}, \mu_1(T_{d-i+1}) + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1}\right\} = \frac{1}{2} + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1}.$$

□

6 Upper Bounds via Intersections

In the proof of [9, Prop. 3.3], Merino and Schymura implicitly use the following, which we will prove explicitly since the second upper bound for covering minima we give will derive from this observation.

Lemma 6.1. *Let $f_1, \ldots, f_d$ be a linear basis of $\mathbb{R}^d$. Then, for every $i \in [d]$ and for every $U \in \mathcal{A}_{d-i}(\mathbb{R}^d)$, there exist $1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant d$ such that*

$$U \cap \operatorname{span}_{\mathbb{R}}\{f_{j_1}, \ldots, f_{j_i}\} \neq \varnothing.$$

Proof. Write down the affine subspace U in the given basis as a translate by the vector $\sum_{j=1}^{d} \alpha_j f_j$ of the span of $d-i$ linearly independent vectors $\sum_{k=1}^{d} \lambda_{l,k} f_k$, for $1 \leqslant l \leqslant d-i$:

$$U = \sum_{j=1}^{d} \alpha_j f_j + \operatorname{span}_{\mathbb{R}} \left\{ \sum_{k=1}^{d} \lambda_{l,k} f_k \mid l \in [d-i] \right\}.$$

If we view this linear subspace as a $(d-i) \times d$ matrix $A = [\lambda_{l,k}]_{l \in [d-i], k \in [d]} \in \mathbb{R}^{(d-i)\times d}$, linear independence of $\sum_{k=1}^{d} \lambda_{l,k} f_k$, for $1 \leqslant l \leqslant d-i$, which are exactly the row vectors of this matrix in the given basis, implies that the rank of this matrix is $d-i$. Therefore, only by using row operations, we can find an equivalent matrix $A' \in \mathbb{R}^{(d-i)\times d}$ which has a $(d-i) \times (d-i)$ submatrix which is up to column swaps, the identity matrix. Without loss of generality, suppose that A' has exactly the $(d-i) \times (d-i)$ identity matrix as its first $d-i$ columns. This assumption corresponds to swapping the indices of the original basis vectors, or in the language the he teorem is stated, its forcing $\{j_1, \ldots, j_i\}$ to be exactly the set $\{d-i+1, \ldots, d\}$. Let

$$A' = \begin{bmatrix} 1 & 0 & \cdots & 0 & \lambda'_{d-i+1,1} & \cdots & \lambda'_{d,1} \\ 0 & 1 & \cdots & 0 & \lambda'_{d-i+1,2} & \cdots & \lambda'_{d,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda'_{d-i+1,d-i} & \cdots & \lambda'_{d,d-i} \end{bmatrix} \in \mathbb{R}^{(d-i)\times d}.$$

Since we got from A to A' just by row operations, their row vectors in the given basis have the same $\mathbb{R}$ span, ie

$$\operatorname{span}_{\mathbb{R}} \left\{ \sum_{k=1}^{d} \lambda_{l,k} f_k \mid l \in [d-i] \right\} = \operatorname{span}_{\mathbb{R}} \left\{ f_l + \sum_{k=d-i+1}^{d} \lambda'_{l,k} f_k \mid l \in [d-i] \right\}.$$

Now, since $U = \sum_{j=1}^{d} \alpha_j f_j + \operatorname{span}_{\mathbb{R}} \left\{ g_l := f_l + \sum_{k=d-i+1}^{d} \lambda'_{l,k} f_k \mid l \in [d-i] \right\}$, let's find one element in $U \cap \operatorname{span}_{\mathbb{R}}\{f_{d-i+1}, \ldots, f_d\}$.

$$U \ni \sum_{j=1}^{d} \alpha_j f_j + \sum_{l=1}^{d-i} (-\alpha_l) g_l = \sum_{j=1}^{d} \alpha_j f_j - \sum_{l=1}^{d-i} \alpha_l \left(f_l + \sum_{k=d-i+1}^{d} \lambda'_{l,k} f_k \right) =$$

$$= \sum_{j=1}^{d} \alpha_j f_j - \sum_{l=1}^{d-i} \alpha_l f_l - \sum_{l=1}^{d-i} \alpha_l \sum_{k=d-i+1}^{d} \lambda'_{l,k} f_k = \sum_{j=d-i+1}^{d} \alpha_j f_j - \sum_{k=d-i+1}^{d} \left(\sum_{l=1}^{d-i} \alpha_l \right) \lambda'_{l,k} f_k =$$

$$= \sum_{k=d-i+1}^{d} \left(\alpha_k - \left(\sum_{l=1}^{d-i} \alpha_l \right) \lambda'_{l,k} \right) f_k \in \operatorname{span}_{\mathbb{R}}\{f_{d-i+1}, \ldots, f_d\}.$$

Therefore, $U \cap \mathrm{span}_{\mathbb{R}}\{f_{d-i+1}, \ldots, f_d\} \neq \varnothing$, ie every $(d-i)$-dimensional affine subspace of $\mathbb{R}^d$ intersects some coordinate i-dimensional subspace in any basis. $\square$

Given this lemma, we know that every affine $d-i$ subspace has to intersect at least one of the i- dimensional coordinate subspaces. Therefore, any set covering all i-dimensional coordinate subspaces has to intersect every affine $d-i$ subspace. We will use this to derive the following upper bound on the i-th covering minimum of any convex body.

Theorem 6.2. *Let $K \in \mathcal{K}^d$, $\Lambda \in \mathcal{L}^d$ and let $\{f_1, \ldots f_d\}$ be a basis of Λ. For $I \in \binom{[d]}{i}$, denote by $L_I = \mathrm{span}_{\mathbb{R}}\{f_i \mid i \in I\}$ the i-dimensional linear subspace of $\mathbb{R}^d$ corresponding to I and the given basis. If for every $I \in \binom{[d]}{i}$, $\dim(K \cap L_I) = i$, then:*

$$\mu_i(K, \Lambda) \leqslant \max\left\{\mu(K \cap L_I, \Lambda \cap L_I) \mid I \in \binom{[d]}{i}\right\}.$$

Proof. Let U be an arbitrary $(d-i)$-dimensional affine subspace of $\mathbb{R}^d$. By Lemma 6.1, there exists $I \in \binom{[d]}{i}$ such that $U \cap L_I \neq \varnothing$. Restricting to L_I, since $K \cap L_I$ is full dimensional and $\Lambda \cap L_i$ is a lattice in L_i since $f_1, \ldots, f_d$ is a lattice basis, by the definition of covering radius:

$$\mu(K \cap L_I, \Lambda \cap L_I)(K \cap L_I) + \Lambda \cap L_I = L_I$$

$$\Rightarrow \mu(K \cap L_I, \Lambda \cap L_I)K + \Lambda \supseteq L_I$$

$$\Rightarrow (\mu(K \cap L_I, \Lambda \cap L_I)K + \Lambda) \cap U \supseteq L_I \cap U \neq \varnothing$$

Let $t \geqslant 0$. The previous calculations imply that for $(tK + \Lambda) \cap U$ to be nonempty, it is enough for t to be greater or equal to $\mu(K \cap L_I, \Lambda \cap L_I)$. Therefore, for $tK + \Lambda$ to intersect all $(d-i)$-dimensional affine subspaces, it is enough that t is the maximum of these values when we iterate $I \in \binom{[d]}{i}$. $\square$

Remark 6.3. Notice that the supposition of all coordinate intersections of dimension i being full dimensional is not that harsh – covering minima are translatory invariant, so for every convex body it suffices to translate it so $\mathbb{0}_d$ is in the interior. Nevertheless, the bound is not translatory invariant and we give it in this level of generality because for example this supposition holds for S_d and the standard basis $e_1, \ldots, e_d$ of $\mathbb{Z}^d$, and is going to be sharp when viewing S_d as is, without translating $\mathbb{0}_d$ into the interior.

6.1 Terminal Simplices

We would like to see what the upper bound we obtained in Theorem 5.3 is for the family of polytopes we are the most interested in – terminal simplices. To get explicit numbers, we need a result due to Codenotti, Santos and Schymura ([5]), where they calculate the covering radii of weighted versions of terminal simplices.

Theorem 6.4. *[5, Theorem 1.4] For every $\omega \in \mathbb{R}^{d+1}_{>0}$, we have:*

$$\mu(S(\omega)) = \frac{\sum\limits_{0 \leqslant i < j \leqslant d} \frac{1}{\omega_i \omega_j}}{\sum\limits_{i=0}^{d} \frac{1}{\omega_i}}$$

Notice that this theorem is a generalization of Theorem 3.4. The proof of this theorem relies on the construction of certain regions, which can be seen as regions induced by a hyperplane arrangement, and furthermore analysing the *alcoved arrangement*, whichs description can be found in [1]. Then, they reduce the problem of finding the covering radius of $S(\omega)$ to the problem of finding a certain last covered point in a cell of this arrangement, which in this case can be reduced to a system of linear equations.

In the next corollary, we give another improvement on the upper bound for the covering minima of terminal simplices.

Corollary 6.5. *For $d \in \mathbb{N}$, $i \in [d]$, the following inequality holds:*

$$\mu_i(T_d) \leqslant \frac{i}{2}\left(1 + \frac{d-i}{d+1}\right)$$

Proof. The goal is to use Theorem 6.2 for the standard basis vectors.
First, notice that T_d is symmetric with respect to the standard basis vectors, so the bound from Theorem 6.2 becomes:

$$\mu_i(T_d) \leqslant \mu(T_d \cap (\mathbb{R}^i \times \{\mathbb{0}_{d-i}\})).$$

Recall that Lemma 5.10 gives us that $T_d \cap (\mathbb{R}^i \times \{\mathbb{0}_{d-i}\}) = S\left(\frac{1}{d-i+1}, 1, \ldots, 1\right) \times \{\mathbb{0}_{d-i}\}$.
Now, from Theorem 6.4 we conclude:

$$\mu\left(S\left(\frac{1}{d-i+1}, 1, \ldots, 1\right)\right) = \frac{(d-i+1)\cdot\binom{i}{1} + 1\cdot\binom{i}{2}}{(d-i+1)\cdot 1 + 1\cdot\binom{i}{1}} = \frac{di - i^2 + i + \frac{i^2-i}{2}}{d+1} =$$

$$= \frac{2di - i^2 + i}{2(d+1)} = \frac{i}{2}\frac{2d-i+1}{d+1} = \frac{i}{2}\left(1 + \frac{d-i}{d+1}\right)$$

□

6.2 Locally anti-blocking Bodies

Our next goal is to find a family of convex bodies for which our bound given in Theorem 6.2 is sharp. The following definition is due to Kohl, Olsen and Sanyal ([17]).

Definition 6.6. A convex body $K \in \mathcal{K}^d$ is called *locally anti-blocking* if for every coordinate subspace L, $\pi_L(K) = K \cap L$.

This family of convex bodies is a generalization of both *anti-blocking bodies* ([8]) mostly known in the context of combinatorial optimization and *unconditional bodies*, which are heavily studied both in the context of convex geometry and functional analysis. For a review of all three of these classes of polytopes, we refer to [17].

Notice that for any convex body, the covering radii of projections to coordinate subspaces give lower bounds, and intersections with the same subspaces give upper bounds for covering minima. Since for locally anti-bocking bodies the projections and intersections are the same, the following corrolary holds.

Corollary 6.7. *Let $K \in \mathcal{K}^d$ be a locally anti-blocking convex body and $i \in [d]$. For $I \in \binom{[d]}{i}$, denote by $L_I = \mathrm{span}_{\mathbb{R}}\{e_i \mid i \in I\}$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. Then:*

$$\mu_i(K) = \max\left\{\mu(K \cap L_I) \mid I \in \binom{[d]}{i}\right\}$$

Proof. By a direct application of Theorem 6.2, we get the wanted upper bound for $\mu_i(K)$. By unconditionallity, for every $I \in \binom{[d]}{i}$, $K \cap L_I = \pi_{L_I}(K)$, and since L_I is a coordinate subspace, $\mathbb{Z}^d \cap L_I = \pi_{L_I}(\mathbb{Z}^d) = \mathbb{Z}^i$. By definition of covering minima via projections to rational subspaces,

$$\mu_i(K) \geqslant \mu(\pi_{L_I}(K), \pi_{L_I}(\mathbb{Z}^d)) = \mu(K \cap L_I),$$

and is therefore greater or equal than the maximum of these values. □

7 Comparing Upper Bounds

The goal of this chapter is to compare the upper bounds we obtained in Theorem 5.3 and 6.2 with the already known upper bound involving successive minima due to Kannan and Lovász, as in Lemma 4.1. We will do this for the few examples of convex bodies where all the covering minima are known, as well as the family of terminal simplices, where it is conjectured that $\mu_i(T_d) = \frac{i}{2}$, and $\frac{i}{2}$ is a known lower bound.

We will refrain from comparing with the bound in Theorem 4.4 due to Kannan and Lovász, since that result is asymptotic in nature and is far off from all of the other bounds presented when comparing on specific examples.

7.1 Unimodular Simplices

As seen in Proposition 2.14, for every $i \in [d]$, $\mu_i(S_d) = i$.

- **Kannan & Lovász bound (4.1)**
  By Examples 1.25 and 1.31, $\lambda_1(S_d - S_d) = \cdots = \lambda_d(S_d - S_d) = 1$ and $\omega(S_d) = 1$. Furthermore, by Lemma 2.6, $\mu_1(S_d) = \frac{1}{\omega(S_d)} = 1$. Therefore, the first bound due to Kannan and Lovász (Lemma 4.1) is sharp for every $i \in [d]$– $i + 1 = \mu_{i+1}(S_d) \leqslant \mu_i(S_d) + \lambda_{d-i}(S_d - S_d) = i + 1$.

- **Intersection bound (5.3)**
  Notice that the unimodular simplex S_d can be seen as the d-fold direct sum of unit length intervals with $\mathbb{0}_d$ on the boundary, ie $S_d = \bigoplus_{j=1}^{d} [\mathbb{0}_d, e_j]$. Therefore, we can make use of Theorem 5.4, which is a corollary of Theorem 5.3. Specifically,
  $$\mu_i(S_d) = \mu_i\left(\bigoplus_{j=1}^{d}[\mathbb{0}_d, e_j], \bigoplus_{j=1}^{d}\mathbb{Z}\right) = \max_{\substack{a_1+\cdots+a_d=i \\ 0\leqslant a_j\leqslant 1}} \sum_{j=1}^{d} \mu_{a_j}\left([\mathbb{0}_d, e_j]\right).$$
  Since $\mu_1([\mathbb{0}_d, e_j]) = 1$ for all $j \in [d]$, and μ_0 is 0 for all convex bodies, the fact that the indices sum up to i implies $\mu_i(S_d) = i$. Therefore, the bound from Theorem 5.3 is also sharp for the unimodular simplex when projections are appropriately chosen – the coordinate axes.

- **Projection bound (6.2)**
  S_d is locally anti-blocking since for every coordinate i-hyperplane L, $\pi_L(S_d) = S_i = S_d \cap L$, therefore by Corollary 6.7, equality is attained in the bound from Theorem 6.2. Moreover, since all coordinate intersections are unimodular simplices of lower dimension, this is a way to calculate all covering minima of all unimodular simplices utilizing the value of covering radii for all unimodular simplices.

7.2 Hypercubes

For a hypercube $C_d = [-1, 1]^d$, we know from Example 2.13 that $\mu_1(C_d) = \cdots = \mu_d(C_d) = \frac{1}{2}$.

- **Kannan & Lovász bound (4.1)**
  Since $C_d - C_d = 2C_d$ and all the successive minima of the hypercube are 1 (see

Example 1.23), $\lambda_1(C_d - C_d) = \cdots = \lambda_d(C_d - C_d) = \frac{1}{2}$. Additionally, from Example 1.38 we see $\mu_1(C_d) = \frac{1}{\omega(C_d)} = \frac{1}{2}$. Therefore, the bound in Lemma 4.1 is not sharp at any step, ie $\frac{1}{2} = \mu_{i+1}(C_d) \leqslant \mu_i(C_d) + \lambda_{d-i}(C_d - C_d) = \frac{1}{2} + \frac{1}{2} = 1$. By successive application of this bound, one would get $\mu_i(C_d) \leqslant \frac{d}{2}$.

- **Intersection bound (5.3)**
Theorem 5.3 is also not of much use in this case. Specifically, if we take a projection to a coordinate j subspace L, it gives
$$\frac{1}{2} = \mu_i(C_d) \leqslant \max_k \mu_k(\pi_L(C_d), \pi_L(\mathbb{Z}^d)) + \mu_{i-k}(C_d \cap L^\perp, \mathbb{Z}^d \cap L^\perp) =$$
$$= \max_k \mu_k(C_j) + \mu_{i-k}(C_{d-j}) = \frac{1}{2} + \frac{1}{2} = 1.$$
If L is not a coordinate subspace, even though the summand coming from the projection becomes much smaller because the lattice gets denser, the summand coming from the intersection gets much bigger and makes the bound worse.

- **Projection bound (6.2)**
The hypercube is again a locally anti-blocking body, therefore the bound from Theorem 6.2 will be sharp. Similarly to the case of the unimodular simplex in Section 7.1, for a i-dimensional coordinate subspace L, $C_d \cap L = C_i$. Therefore, this bound reduces the calculation of all covering minima of hypercubes of all dimensions to calculating covering radii of all hypercubes.

7.3 Crosspolytopes

- **Kannan & Lovász bound (4.1)**
As already discussed in Example 4.2, for every $i \in [d]$, $\mu_i(C_d^*) = \frac{d}{2}$, and the successive minima bound is sharp in this case.

- **Intersection bound (5.3)**
Notice that we can see the crosspolytope C_d^* as a d-fold direct sum of intervals with $\mathbb{0}_d$ in their relative interiors, ie $C_d^* = \bigoplus_{j=1}^{d}[-e_j, e_j]$. Similar as in the case of the unimodular simplices, we can now make use of Theorem 5.3 through the formula for the covering minima of direct sums in Theorem 5.4, which gives us:
$$\mu_i(C_d^*) = \mu_i\left(\bigoplus_{j=1}^{d}[-e_j, e_j], \bigoplus_{j=1}^{d}\mathbb{Z}\right) = \max_{\substack{a_1+\cdots+a_d=i \\ a_j\in\{0,1\}}} \mu_{a_j}\left([-e_j, e_j]\right).$$

Since $\mu_1([-e_j, e_j]) = \frac{1}{2}$ for all $j \in [d]$, and the indices sum up to i, it follows that $\mu_i(C_d^*) = \frac{d}{2}$. One could notice that while using this bound to calculate $\mu_i(C_d^*)$, we did not use the symmetry of the crosspolytope to its fullest extent. Specifically, this whole calculation would work for any direct sum of intervals of length 2 in their corresponding coordinate axes, containing $\mathbb{0}_d$ in its interior. Moreover, $\mathbb{0}_d$ being in the interior or boundary of the segments does not matter at all – if it is in the interior of all segments, we get a polytope resembling the crosspolytope, and if its on the boundary of every segment we get a simplex with d orthogonal edges, and all other options interpolate between the two. The lengths of the segments do not matter that much either, for segments of lenghts $l_1 \leqslant \cdots \leqslant l_d$, the i-th covering minimum of their direct sum would be $\sum_{j=1}^{i} \frac{1}{l_j}$. One could also notice that here we did not use the fact that the covering radius of a crosspolytope is half of its dimension, which as seen in Example 1.42 requires a bit of calculation itself.

- **Projection bound (6.2)**
  As in the previous two cases, the crosspolytope is also a locally anti-blocking body, which when intersected with a coordinate subspace gives a crosspolytope of the appropriate dimension. Therefore, the intersection bound is also sharp, and can generate all covering minima of a crosspolytope from knowing the covering radii of crosspolytopes in lower dimensions.

7.4 $P_{d,i}$

Recall from Section 2.2 for $i \in [d]$ the family of polytopes $P_{d,i} := C_d \cap iC_d^*$, for which Merino and Schymura ([9]) calculated all covering minima to be:

$$\mu_j(P_{d,i}) = \begin{cases} \frac{1}{2} & , j \leqslant i \\ \frac{j}{2i} & , j > i. \end{cases}$$

Since for $i = 1$ and $i = d$, $P_{d,i}$ is respectively C_d^* and C_d, which we already analized, we will now focus on the cases $2 \leqslant i \leqslant d$

- **Kannan & Lovász bound (4.1)**

  Observe that $C_d^* \subseteq P_{d,i} \subseteq C_d$, therefore for every $j \in [d]$, $\lambda_j(C_d^*) \geqslant \lambda_j(P_{d,i}) \geqslant \lambda_j(C_d)$. Since all the successive minima of C_d and C_d^* are 1, so are the successive minima of $P_{d,i}$. Therefore, $\lambda_j(P_{d,i} - P_{d,i}) = \lambda_j(2P_{d,i}) = \frac{1}{2}$.

  The Kannan and Lovász bound is not tight for the first $i - 1$ covering minima: $\frac{1}{2} = \mu_{j+1}(P_{d,i}) \leqslant \mu_j(P_{d,i}) + \lambda_{d-j}(P_{d,i} - P_{d,i}) = \frac{1}{2} + \frac{1}{2} = 1$. Moreover, since $i > 1$, this bound is also not tight for any $j > i : \frac{j+1}{2i} = \mu_{j+1}(P_{d,i}) \leqslant \mu_j(P_{d,i}) + \lambda_{d-j}(P_{d,i} - P_{d,i}) = \frac{j}{2i} + \frac{1}{2} = \frac{j+i}{2i}$.

- **Projection bound (5.3)**

  Notation 7.1. For convenience, for $l < i$ define $P_{l,i} = C_l \cap iC_l^* = C_l$.

  Notice that for a j-dimensional coordinate subspace L_j,

  $$\pi_{L_j}(P_{d,i}) = P_{d,i} \cap L_j = C_d \cap iC_d^* \cap L_j = C_j \cap iC_j^* = P_{j,i}.$$

  Therefore, Theorem 5.3 for $P_{d,i}$ with respect to the subspace L_j gives:

  $$\mu_l(P_{d,i}) \leqslant \max_k \mu_k(P_{j,i}) + \mu_{l-k}(P_{d-j,i}).$$

  Our goal now is to see in which cases an appropriate choice of j can give us the tight bound.
  If $l \leqslant i$, the left hand side is $\frac{1}{2}$, and however we choose $1 \leqslant j \leqslant d-1$, there will exist a k on the right hand side such that $\mu_k(P_{j,i}), \mu_{l-k}(P_{d-j,i}) \neq 0$. Since both of these values are at least $\frac{1}{2}$, the best we could hope to get in this case is $\mu_l(P_{d,i}) \leqslant 1$, which is not sharp.
  If $l > i$, $\mu_l(P_{d,i}) = \frac{l}{2i}$. Notice that if for all viable k on the right hand side, $k \geqslant i$ and $l - k \geqslant i$, we would get $\frac{k}{2i} + \frac{l-k}{2i} = \frac{l}{2i}$ on the right hand side, ie the tight upper bound. If these two inequalities are not satisfied, this bound will not be tight since one of the covering minima would be $\frac{1}{2}$, which would in that case be strictly bigger than $\frac{k}{2i}$ or $\frac{l-k}{2i}$. Now, let's see when these two inequalities can be satisfied for all k. Firstly, $\max\{0, l+j-d\} \leqslant k \leqslant \min\{j, l\}$ are all the k that we should take into consideration in Theorem 5.3. The inequalities we want to hold ammount to $i \leqslant k \leqslant l-i$. Therefore, it would suffice if $i \leqslant \max\{0, l+j-d\}$ and $\min\{j, l\} \leqslant l-i$. This is equivalent to $l + j \geqslant d$, $j \leqslant l$ and $d - (l-i) \leqslant j \leqslant l - i$, where we see that the first two inequalities are redundant. Therefore, the appropriate j such that the projection bound onto a coordinate j subspace would give us the sharp upper bound for $\mu_l(P_{d,i})$ would be any $d - (l-i) \leqslant j \leqslant l - i$. Notice that such a j exists if and only if $l - i \geqslant \frac{d}{2}$.

- **Intersection bound (6.2)**
  Concerning our intersection bound – $P_{d,i}$ are locally anti-blocking bodies, so by Corollary 6.7, it is always tight. Specifically, the proof of this Corollary actually mimicks the calculation of the covering minima of $P_{d,i}$ in [9].

The following table summarizes the previous comparisons.

	Kannan & Lovász (4.1)	Projection bound (5.3)	Intersection bound (6.2)
S_d	tight	tight	tight
C_d	not tight, $\frac{d-1}{2}$ off	not tight, $\frac{1}{2}$ off	tight
C_d^*	tight	tight	tight
$P_{d,i}$	not tight	tight in dimensions $\geqslant i + \frac{d}{2}$	tight

7.5 Terminal Simplices

Regarding Terminal Simplices, we don't know the values of their covering minima, therefore we can only compare upper bounds amongst each other instead of talking about when they are thight. The bounds that we will take into consideration are:

1. $\boldsymbol{\mu_i(T_d) \leqslant i}$.
   This bound comes from the fact that the i-th covering minimum is the maximum of covering radii of i-dimensional rational projections, the fact that lattice polytopes project to lattice polytopes, and Proposition 3.1.

2. $\boldsymbol{\mu_i(T_d) \leqslant \frac{d}{2}}$.
   This bound comes from the fact that covering minima are monotone with respect to the index, therefore $\mu_i(T_d) \leqslant \mu_d(T_d) = \mu(T_d)$, and the latter is equal to $\frac{d}{2}$ as in Theorem 3.4.

3. $\boldsymbol{\mu_i(T_d) \leqslant \frac{1}{2} + \frac{(i-1)d}{d+1} =: B^{KL}_{d,i}}$.

   This bound is obtained by successive use of Lemma 4.1 and the fact that $\mu_1(T_d) = \frac{1}{2}$, together with the fact that $\lambda_i(T_d - T_d) = \frac{d}{d+1}$ for all $i \in [d]$ (see [4], proof of Proposition 3.12).

4. $\boldsymbol{\mu_i(T_d) \leqslant \frac{1}{2} + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1} =: B^{\pi}_{d,i}}$
   This is the bound from Corrolary 5.12, which was obtained using the projection bound from Theorem 5.3.

5. $\boldsymbol{\mu_i(T_d) \leqslant \frac{i}{2}\left(1 + \frac{d-i}{d+1}\right) =: B^{\cap}_{d,i}}$
   This is the bound from Corrolary 6.5, which was obtained using the intersection bound from Theorem 6.2.

- **Comparing i and $B^{\cap}_{d,i}$:**
  We can calculate $B^{\cap}_{d,i} = \frac{i}{2}\left(1 + \frac{d-i}{d+1}\right) = \frac{i}{2} + \frac{i(d-i)}{2(d+1)} = i - \frac{i(i+1)}{2(d+1)}$ and see that this is an improvement of the $\mu_i(T_d) \leqslant i$ bound by a linear factor of $\frac{i(i+1)}{2(d+1)}$. We can also notice that this improvement is significantly better for bigger values of i then for smaller ones.

- **Comparing $\frac{d}{2}$ and $B^{\cap}_{d,i}$:**
  Calculating $B^{\cap}_{d,i} = \frac{i}{2} + \frac{i(d-i)}{2(d+1)} = \frac{d}{2} + \frac{i(d-i)-(d-i)(d+1)}{2(d+1)} = \frac{d}{2} - \frac{(d-i)(d+1-i)}{2(d+1)}$, we see that $B^{\cap}_{d,i}$ is better by a linear factor of $\frac{(d-i)(d+1-i)}{2(d+1)}$, which is strictly positive for all $i < d$. This improvement is significantly better for smaller values of i than for bigger ones, but that is because the bound of $\frac{d}{2}$ is sharp for $i = d$ and doesn't get better by varying the parameter i.

- **Comparing i and $B^{\pi}_{d,i}$:**
  By calculating $B^{\pi}_{d,i} = \frac{1}{2} + \sum_{j=0}^{i-2} \frac{d-j}{d-j+1} = \frac{1}{2} + i - 1 - \sum_{j=0}^{i-2} \frac{1}{d-j+1} = i - \left(\frac{1}{2} + \sum_{j=0}^{i-2} \frac{1}{d-j+1}\right)$, we see that $B^{\pi}_{d,i}$ is always better.

- **Comparing $\frac{d}{2}$ and $B^{\pi}_{d,i}$:**
  To compare $B^{\pi}_{d,i}$ and $\mu_i(T_d) \leqslant \mu(T_d) = \frac{d}{2}$, we calculate $B^{\pi}_{d,i} = \frac{1}{2} + \sum_{j=0}^{i-2} \frac{(d-j+1)+(d-j-1)}{2(d-j+1)} = \frac{i}{2} + \sum_{j=0}^{i-2} \frac{d-j-1}{2(d-j+1)} = \frac{d}{2} - \left(\frac{d-i}{2} - \sum_{j=0}^{i-2} \frac{d-j-1}{2(d-j+1)}\right)$. Notice that for a fixed d, $\frac{d-i}{2} - \sum_{j=0}^{i-2} \frac{d-j-1}{2(d-j+1)}$ is a monotonely decreasing function of i, and that it is positive for $i = 1$ and negative for $i = d-1$. Therefore, for i small with respect to d, the bound $B^{\pi}_{d,i}$ is better than the bound $\frac{d}{2}$, and worse for i big with respect to d.

- **Comparing $B^{KL}_{d,i}$ and $B^{\cap}_{d,i}$:**

  We want to see for which pairs of d and i is $B^{\cap}_{d,i} \leqslant B^{KL}_{d,i}$. Let's transform $B^{KL}_{d,i}$ so it's more convenient to compare to $B^{\cap}_{d,i}$ in the given form. $B^{KL}_{d,i} = \frac{1}{2} + \frac{(i-1)d}{d+1} = \frac{1}{2} + \frac{(i-1)(d+1)}{2(d+1)} + \frac{(i-1)(d-1)}{2(d+1)} = \frac{i}{2} + \frac{(i-1)(d-1)}{2(d+1)} = \frac{i}{2}\left(1 + \frac{(i-1)(d-1)}{i(d+1)}\right)$. Therefore, $B_{\cap} \leqslant B_{KL}$ if and only if $\frac{d-i}{d+1} \leqslant \frac{(i-1)(d-1)}{i(d+1)}$, ie $i(d-i) \leqslant (i-1)(d-1) = (i-1)(d-i) + (i-1)^2$. Subtracting $(i-1)(d-1)$ from both sides of this inequality, we get $d - i \leqslant (i-1)^2$. Transforming this a bit more, we see that it's equivalent to $(i - \frac{1}{2})^2 \geqslant d - \frac{3}{4}$. Finally, we can conclude that $B^{\cap}_{d,i}$ is better than the Kannan and Lovász bound if and only if $i \geqslant \sqrt{d - \frac{3}{4}} + \frac{1}{2}$.

- **Comparing $B^{KL}_{d,i}$ and $B^{\pi}_{d,i}$:**
  Since for every $1 \leqslant j \leqslant i-2$, $\frac{d-j}{d-j+1} < \frac{d}{d+1}$, we can conclude that $B^{\pi}_{d,i} < B^{KL}_{d,i}$ for all all d and $i > 2$, and we see that the bounds are the same for $i = 2$.

- **Comparing $B^{\pi}_{d,i}$ and $B^{\cap}_{d,i}$:**
  Since $B^{\pi}_{d,i} < B^{KL}_{d,i}$ holds for all d and i, and there exist cases when $B^{\cap}_{d,i} > B^{KL}_{d,i}$, we know at least in those cases that $B^{\pi}_{d,i} < B^{\cap}_{d,i}$. However, for bigger values of i, the $B^{\cap}_{d,i}$ will be better than $B^{\pi}_{d,i}$. We will not provide in closed form when exactly this happens, but since we have shown that for all pairs of d and i $B^{\cap}_{d,i} < \frac{d}{2}$, and that if i is big enough compared to d, $B^{\pi}_{d,i} > \frac{d}{2}$, this implies that in those cases $B^{\cap}_{d,i} < B^{\pi}_{d,i}$. This fact can be also seen by noticing that $B^{\cap}_{d,i} - B^{\cap}_{d,i-1} = \frac{i}{d+1} < \frac{d-i}{d-i+1} = B^{\pi}_{d,i} - B^{\pi}_{d,i}$, for all $i < d$, and even though the $B^{\pi}_{d,i}$ starts off better for small i, this difference is big enough to push the $B^{\pi}_{d,i}$ over $B^{\cap}_{d,i}$ for big i with respect to d.

In the following table we present the calculations of the conjectured values $\frac{i}{2}$, Kannan and Lovaśzes $B^{KL}_{d,i}$ and our bounds $B^{\pi}_{d,i}$ and $B^{\cap}_{d,i}$ for the covering minima of the simplex T_{1000}.

bound \ i	15	45	46	100	500	700	800	999
$\frac{i}{2}$	7.50	22.50	23.00	50.00	250.00	350.00	400.00	499.50
$B^{KL}_{1000,i}$	13.99	43.96	44.96	98.90	498.50	698.30	798.20	997.00
$B^{\pi}_{1000,i}$	13.99	43.96	44.95	98.90	498.31	697.80	797.40	992.35
$B^{\cap}_{1000,i}$	14.88	43.97	44.92	94.96	374.88	454.90	479.92	500.00

Table 1: Values of bounds on $\mu_i(T_{1000})$, rounded up to 2 digits

From Table 7.5 we can notice that the improvement of $B^{\pi}_{d,i}$ compared to $B^{KL}_{d,i}$ is not significant. Nevertheless, we have proved that it will always be a better bound. We can also notice that for $d = 1000$, the $B^{\cap}_{d,i}$ becomes the best bound for $i = 46$, and becomes significantly better for bigger i, being just around $\frac{1}{2}$ away from the lower bound that is conjectured to be sharp for $i = 999$. In general, $B^{\cap}_{d,d-1} - \frac{d-1}{2} = \frac{d-1}{2(d+1)}$, therefore this bound proves that the value of the $(d-1)$-st covering minimum of T_d is in an interval smaller than $\frac{1}{2}$.

7.6 General comparisons

Throughout this section, we highlight some important differences between the three upper bounds on covering minima. Firstly, the bound due to Kannan and Lovász depends on the successive minima of the difference body. Moreover, looking at the proof of Lemma 4.1, we do not see a class of convex bodies for which this bound will be sharp.

Our projection bound depends on calculating the projection and intersection of the convex body with respect to the chosen subspace and its orthogonal complement, and the covering minima of those convex bodies. The positive aspect of this is that it does not depend on successive minima, as the recursive connection between covering minima seems more natural. However, there does not exist an algorithm for calculating covering minima of rational polytopes. As seen in Theorem 5.4, this bound will be sharp for all convex bodies that can be decomposed into a direct sum. However, the result presented there does still depend on smaller covering minima of convex bodies of smaller dimension, which are again hard to calculate, but it is effective in the sense of it's reducing a problem into the same problem in smaller dimensions. We do not know when exactly this bound will be sharp, but we assume that this bound will be close to optimal when the convex body can be well aproximated with a direct sum of ”simpler” bodies. We would also like to point out that the generality in which this bound is given, leaves a lot of degrees of freedom in the choice of which subspaces the bound will emphasise.

As for our intersection bound, it is the best one in the sense of dependencies – it depends only on intersections of the convex body with coordinate subspaces, and the covering radii of those. This is a positive, since there exists an algorithm for calculating the covering radius of rational polytopes. As seen in Corollary 6.7, this bound will be tight for all locally anti-blocking bodies, but the equality case might not be limited to just that family. It is also worth mentioning that this bound is tight for all convex bodies whose covering minima are already known.

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